DECOMPOSITION NUMBERS FOR HECKE ALGEBRAS OF TYPE $G(r,p,n)$: THE (ε,q)-SEPARATED CASE

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ABSTRACT. The paper studies the modular representation theory of the cyclotomic Hecke algebras of type $G(r,p,n)$ with (ε,q)-separated parameters. We show that the decomposition numbers of these algebras are completely determined by the decomposition matrices of related cyclotomic Hecke algebras of type $G(s,1,m)$, where $1 \le s \le r$ and $1 \le m \le n$. Furthermore, the proof gives an explicit algorithm for computing these decomposition numbers. Consequently, in principle, the decomposition matrices of these algebras are now known in characteristic zero.

In proving these results, we develop a Specht module theory for these algebras, explicitly construct their simple modules and introduce and study analogues of the cyclotomic Schur algebras of type $G(r,p,n)$ when the parameters are (ε,q)-separated.

The main results of the paper rest upon two Morita equivalences: the first reduces the calculation of all decomposition numbers to the case of the l-*splittable decomposition numbers* and the second Morita equivalence allows us to compute these decomposition numbers using an analogue of the cyclotomic Schur algebras for the Hecke algebras of type $G(r,p,n)$.

CONTENTS

1. INTRODUCTION

The cyclotomic Hecke algebras [6] are an important class of algebras which arise in the representation theory of finite reductive groups. These algebras can be defined using generators and relations and they are deformations of the group algebras of the complex reflection groups. The cyclotomic Hecke algebras can also be constructed using the monodromy representation of the associated braid groups [7] and,

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in characteristic zero, they are closely connected with category $\mathcal{O}$ for the rational Cherednik algebras by the Knizhnik-Zamolodchikov functor [19].

This paper is concerned with the representation theory of the cyclotomic Hecke algebras $\mathscr{H}_{r,p,n}$ of type $G(r,p,n)$, where $r = pd$, $p > 1$ and $n \geq 3$. Throughout we work over a field K which contains a primitive pth root of unity ε. The algebra $\mathscr{H}_{r,p,n}$ depends upon the parameters $q \in K$ and $\mathbf{Q} = (Q_1, \dots, Q_d) \in K^d$ (see Definition 2.2). The d-tuple of parameters $\mathbf{Q}$ is **(ε, q)-separated** over K if

$$\prod_{1\le i,j\le d}\ \prod_{-n<k<n}\ \prod_{1\le t<p} \big(Q_i - \varepsilon^t q^k Q_j\big) \neq 0. \tag{1.1}$$

As we explain in Lemma 2.6, (ε, q)-separation is almost the same as assuming that $\langle\varepsilon\rangle \cap \langle q\rangle = 1$. This can be viewed as the quantum analogue of the common assumption in Clifford theory that the characteristic of the field should not divide the index of the normal subgroup inside the parent group. In general, the algebra $\mathscr{H}_{r,p,n}$ is not semisimple when $\mathbf{Q}$ is (ε, q)-separated.

The main result of this paper is the following.

Theorem A. *Suppose that K is a field of characteristic zero and that $\mathbf{Q}$ is (ε, q)-separated over K. Then the decomposition matrix of $\mathscr{H}_{r,p,n}$ is determined by the decomposition matrices of the cyclotomic Hecke algebras of type $G(s,1,m)$, where $1 \le s \le r$ and $1 \le m \le n$.*

In proving this result we also obtain an analogous but slightly weaker result for the decomposition numbers of $\mathscr{H}_{r,p,n}$ in positive characteristic. Moreover, when combined with the results of [25], Theorem A gives an explicit algorithm for computing the decomposition numbers of $\mathscr{H}_{r,p,n}$ in terms of the decomposition matrices of related Hecke algebras of type $G(s,1,m)$. Ariki [3] has determined the decomposition numbers of the Hecke algebras $\mathscr{H}_{r,n} = \mathscr{H}_{r,1,n}$ of type $G(r,1,n)$ when he, famously, proved and generalised the LLT conjecture. Hence, combining [3] and Theorem A implies the following.

Corollary. *Suppose that K is a field of characteristic zero and that $\mathbf{Q}$ is (ε, q)-separated over K. Then the decomposition matrix of $\mathscr{H}_{r,p,n}$ is, in principle, known.*

We note that Theorem A and its corollary have been obtained by the first author in the special case of the Hecke algebras of type D, when $r = p = 2$ [24]. This paper is a (non-trivial) generalization of the results in [24] to the algebras $\mathscr{H}_{r,p,n}$.

To prove Theorem A it is enough by [25, Theorem B] (see Theorem 2.4), to compute the l–**splittable** decomposition numbers of the Hecke algebras of type $G(r,p,n)$. As is usual in Clifford theory, a decomposition number $[S : D]$ for $\mathscr{H}_{r,p,n}$ is p**-splittable** if S and D both have trivial *inertia groups*; see Definition 1.5 for a purely combinatorial definition.

In Theorem D below we give a closed formula for all of the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$. This formula depends on the decomposition numbers of certain Hecke algebras $\mathscr{H}_{s,m} = \mathscr{H}_{s,1,m}$, where $s \le r$ and $m \le n$, and some scalars $\mathfrak{g}_{\boldsymbol{\lambda}} \in K$ which come from the semisimple representation theory of $\mathscr{H}_{s,m}$. More precisely, $\mathfrak{g}_{\boldsymbol{\lambda}}$ is an lth root of a quotient of two **Schur elements**. The scalars $\mathfrak{g}_{\boldsymbol{\lambda}}$ enter the picture because they can be used to decompose the Specht modules of $\mathscr{H}_{r,n}$ into a direct sum of $\mathscr{H}_{r,p,n}$-modules.

All of the results in this paper are geared towards computing the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$. This requires a considerable amount of preliminary

work, much of which takes place inside the algebra $\mathscr{H}_{r,n}$. This story begins with the Morita equivalence theorem of Dipper and the second author [12] which shows, that modulo some technical assumptions on $\mathbf{Q}$, that there is a Morita equivalence

$$\text{Mod-}\mathscr{H}_{r,n} \xrightarrow[\text{Morita}]{\simeq} \bigoplus_{\mathbf{b}\in\mathscr{C}_{p,n}} \text{Mod-}\mathscr{H}_{d,\mathbf{b}}, \tag{1.2}$$

where $\mathscr{C}_{p,n}$ is the set of compositions of n into p parts and if $\mathbf{b} = (b_1, \dots, b_p) \in \mathscr{C}_{p,n}$ then $\mathscr{H}_{d,\mathbf{b}} = \mathscr{H}_{d,b_1} \otimes \cdots \otimes \mathscr{H}_{d,b_p}$. This result is proved by constructing an explicit $(\mathscr{H}_{d,\mathbf{b}}, \mathscr{H}_{r,n})$-bimodule $V_{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ (Definition 2.12), and showing that $V_{\mathbf{b}}$ is projective as an $\mathscr{H}_{r,n}$-module and that $\mathscr{H}_{d,\mathbf{b}} \cong \operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}})$.

In this paper we use the Morita equivalence (1.2) to understand how the Specht modules of $\mathscr{H}_{r,n}$ behave under restriction to $\mathscr{H}_{r,p,n}$. One of the key results is Theorem 2.31 which shows that there is an invertible central element $z_{\mathbf{b}}$ in $\mathscr{H}_{d,\mathbf{b}}$ such that $e_{\mathbf{b}} = z_{\mathbf{b}}^{-1} \cdot v_{\mathbf{b}} T_{\mathbf{b}}$ is the idempotent in $\mathscr{H}_{r,n}$ which generates $V_{\mathbf{b}}$, where $T_{\mathbf{b}} = T_{w_{\mathbf{b}}}$ for a certain permutation $w_{\mathbf{b}} \in \mathfrak{S}_n$. As a byproduct we construct a *parabolic subalgebra* of $\mathscr{H}_{r,n}$ which is isomorphic to $\mathscr{H}_{d,\mathbf{b}}$ and we show that the Morita equivalence (1.2) corresponds to induction from these subalgebras.

The first aim of this paper is to show that $z_{\mathbf{b}}$ acts as multiplication by an invertible scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$ on certain Specht modules of $\mathscr{H}_{r,n}$. In order to describe these results, and how they help prove Theorem A, we need some more notation. Recall from [11] that $\mathscr{H}_{r,n}$ is a cellular algebra with cell modules, the **Specht modules** $S(\boldsymbol{\lambda})$, indexed by the r-multipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ of n. If $\mathscr{H}_{r,n}$ is semisimple then the Specht modules are a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{r,n}$-modules. More generally, define $D(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda})/\operatorname{rad} S(\boldsymbol{\lambda})$, where $\operatorname{rad} S(\boldsymbol{\lambda})$ is the radical of the bilinear form on $S(\boldsymbol{\lambda})$. Then the non-zero $D(\boldsymbol{\lambda})$ are a complete set of pairwise non-isomorphic $\mathscr{H}_{r,n}$-modules.

For each $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$, we write $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{[1]}, \cdots, \boldsymbol{\lambda}^{[p]})$, where

$$\boldsymbol{\lambda}^{[t]} = (\boldsymbol{\lambda}^{(dt-d+1)}, \boldsymbol{\lambda}^{(dt-d+2)}, \dots, \boldsymbol{\lambda}^{(dt)}), \quad \text{for } 1 \le t \le p. \tag{1.3}$$

For convenience, set $\boldsymbol{\lambda}^{[t+kp]} = \boldsymbol{\lambda}^{[t]}$, for all $k \in \mathbb{Z}$. Let $\mathbf{b} = (b_1, \dots, b_p) \in \mathscr{C}_{p,n}$ and set $\mathscr{P}_{d,\mathbf{b}} = \{\, \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \mid |\boldsymbol{\lambda}^{[t]}| = b_t \text{ for } 1 \le t \le p \,\}$. Then, by [11], the algebra $\mathscr{H}_{d,\mathbf{b}}$ is a cellular algebra with cell modules $S_{\mathbf{b}}(\boldsymbol{\lambda}) \cong S(\boldsymbol{\lambda}^{[1]}) \otimes \cdots \otimes S(\boldsymbol{\lambda}^{[p]})$, for $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Again, the modules $D_{\mathbf{b}}(\boldsymbol{\lambda}) = S_{\mathbf{b}}(\boldsymbol{\lambda})/\operatorname{rad} S_{\mathbf{b}}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, are either absolutely irreducible or zero.

Let $\mathcal{F} = \mathbb{Q}(\dot\varepsilon, \dot q, \dot{\mathbf{Q}})$, where $\dot\varepsilon \in \mathbb{C}$ is a fixed primitive pth root of unity in $\mathbb{C}$ and $\dot q$ and $\dot{\mathbf{Q}}$ are indeterminates. The cyclotomic Hecke algebras $\mathscr{H}_{r,n}^{\mathcal{F}}$ and $\mathscr{H}_{d,\mathbf{b}}^{\mathcal{F}}$ over $\mathcal{F}$ are semisimple and they come equipped with non-degenerate trace forms Tr and $\mathrm{Tr}_{\mathbf{b}}$, respectively. Define the **Schur elements** $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}$ and $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}^{\mathbf{b}}$ of $\mathscr{H}_{r,n}^{\mathcal{F}}$ and $\mathscr{H}_{d,\mathbf{b}}^{\mathcal{F}}$, respectively, are the scalars in $\mathcal{F}$ determined by

$$\mathrm{Tr} = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{r,n}} \frac{1}{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}} \chi^{\boldsymbol{\lambda}} \quad \text{and} \quad \mathrm{Tr}_{\mathbf{b}} = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}} \frac{1}{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}^{\mathbf{b}}} \chi_{\mathbf{b}}^{\boldsymbol{\lambda}}, \tag{1.4}$$

where $\chi^{\boldsymbol{\lambda}}$ and $\chi_{\mathbf{b}}^{\boldsymbol{\lambda}}$ are the characters of the irreducible Specht modules $S(\boldsymbol{\lambda})$ and $S_{\mathbf{b}}(\boldsymbol{\lambda})$, respectively.

The Schur elements $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}$ and $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}^{\mathbf{b}}$ are explicitly known [28] and, as we now explain, they are closely related to the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$ which give the action of $z_{\mathbf{b}}$ on the Specht modules of $\mathscr{H}_{r,n}$. To state this result define $\mathsf{o}_{\boldsymbol{\lambda}} = \min\{\, k \ge 1 \mid \boldsymbol{\lambda}^{[k+t]} = \boldsymbol{\lambda}^{[t]}, \text{ for all } t \in \mathbb{Z} \,\}$, and set $p_{\boldsymbol{\lambda}} = p/\mathsf{o}_{\boldsymbol{\lambda}}$. Note that $\mathsf{o}_{\boldsymbol{\lambda}}$ divides p so that $p_{\boldsymbol{\lambda}}$ is an integer, for all $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$.

Theorem B. *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated over* K *and that* $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$*. Then there exists a non-zero scalar* $\mathfrak{f}_{\boldsymbol{\lambda}} \in K$ *such that* $z_{\mathbf{b}} \cdot v = \mathfrak{f}_{\boldsymbol{\lambda}} v$*, for all* $v \in S(\boldsymbol{\lambda})$*. Moreover,*

$$\mathfrak{f}_{\boldsymbol{\lambda}} = (\mathfrak{s}_{\boldsymbol{\lambda}}/\mathfrak{s}^{\mathbf{b}}_{\boldsymbol{\lambda}}) \operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}) = \varepsilon^{\frac{1}{2} d o_{\boldsymbol{\lambda}} n(1-p_{\boldsymbol{\lambda}})} \mathfrak{g}_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}},$$

where $\mathfrak{g}_{\boldsymbol{\lambda}} \in K$ *and where* $(\mathfrak{s}_{\boldsymbol{\lambda}}/\mathfrak{s}^{\mathbf{b}}_{\boldsymbol{\lambda}})(\varepsilon, q, \mathbf{Q}) = (\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}/\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}})(\varepsilon, q, \mathbf{Q})$ *is the specialization of the rational function* $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}/\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}$ *at* $(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}}) = (\varepsilon, q, \mathbf{Q})$ (*which is well-defined and non-zero*).

Roughly half of this paper is devoted to proving Theorem B, but the payoff is considerable as the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$ and $\mathfrak{g}_{\boldsymbol{\lambda}}$ play a role in everything that follows. The three main steps in its proof are Theorem 2.31, which explicitly relates the primitive idempotents in $\mathscr{H}^{\mathcal{F}}_{r,n}$ and $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ under (1.2); Theorem 2.35, which is a comparison theorem relating the trace functions Tr and $\mathrm{Tr}_{\mathbf{b}}$; and, Theorem 3.37, which uses *shifting homomorphisms*, some Clifford theory and seminormal forms to show that $\mathfrak{f}_{\boldsymbol{\lambda}}$ has a $p_{\boldsymbol{\lambda}}$th root.

The reason why Theorem B is important is that multiplication by $z_{\mathbf{b}}$ induces an $\mathscr{H}_{r,n}$-module endomorphism of $S(\boldsymbol{\lambda})$. We show that the factorisation of $\mathfrak{f}_{\boldsymbol{\lambda}}$ given in Theorem B corresponds to a factorisation of this endomorphism and hence that there exists a $\mathscr{H}_{r,p,n}$-module endomorphism $\theta_{\boldsymbol{\lambda}}$ of $S(\boldsymbol{\lambda})$ such that $\theta_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}}$ is $\mathfrak{g}_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}}$ times the identity map on $S(\boldsymbol{\lambda})$ (Corollary 3.46). This allows us to decompose the Specht module as $S(\boldsymbol{\lambda}) = S^{\boldsymbol{\lambda}}_1 \oplus \cdots \oplus S^{\boldsymbol{\lambda}}_{p_{\boldsymbol{\lambda}}}$, where

$$S^{\boldsymbol{\lambda}}_t = \{\, x \in S(\boldsymbol{\lambda}) \mid \theta_{\boldsymbol{\lambda}}(x) = \varepsilon^{t o_{\boldsymbol{\lambda}}} \mathfrak{g}_{\boldsymbol{\lambda}} x \,\}$$

is one of the $\theta_{\boldsymbol{\lambda}}$-eigenspace of $S(\boldsymbol{\lambda})$, for $1 \le t \le p_{\boldsymbol{\lambda}}$. The module $S^{\boldsymbol{\lambda}}_t$ is an $\mathscr{H}_{r,p,n}$-module which is an analogue of a Specht module for $\mathscr{H}_{r,p,n}$.

Next we want to construct the irreducible $\mathscr{H}_{r,p,n}$-modules. Let $D^{\boldsymbol{\lambda}}_t$ be the head of $S^{\boldsymbol{\lambda}}_t$. We will show that $D^{\boldsymbol{\lambda}}_t$ is either irreducible or zero. To describe the complete set of irreducible $\mathscr{H}_{r,p,n}$-modules let $\mathscr{K}_{r,n} = \{\boldsymbol{\lambda} \in \mathscr{P}_{r,n} | D(\boldsymbol{\lambda}) \ne 0\}$ be the set of **Kleshchev multipartitions** for $\mathbf{Q}^{\vee\varepsilon}$. Then $\{\, D(\boldsymbol{\lambda}) \mid D(\boldsymbol{\lambda}) \ne 0 \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{r,n}$-modules by [11]. Define an equivalence relation $\sim_\sigma$ on $\mathscr{P}_{r,n}$ by $\boldsymbol{\lambda} \sim_\sigma \boldsymbol{\mu}$ if there exists a $k \in \mathbb{Z}$ such that $\boldsymbol{\lambda}^{[t]} = \boldsymbol{\mu}^{[t+k]}$, for $1 \le t \le p$ and $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{r,n}$. If $\mathbf{Q}$ is (ε, q)-separated over K, then $\sim_\sigma$ induces an equivalence relation on $\mathscr{K}_{r,n}$ (cf. Lemma 3.3). Let $\mathscr{P}^{\sigma}_{r,n}$ and $\mathscr{K}^{\sigma}_{r,n}$ be the sets of $\sim_\sigma$-equivalence classes in $\mathscr{P}_{r,n}$ and $\mathscr{K}_{r,n}$, respectively.

Theorem C. *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated over the field* K*. Then:*

a) $\{\, D^{\boldsymbol{\mu}}_t \mid \boldsymbol{\mu} \in \mathscr{K}^{\sigma}_{r,n}$ *and* $1 \le t \le p_{\boldsymbol{\mu}} \,\}$ *is a complete set of pairwise non-isomorphic absolutely irreducible* $\mathscr{H}_{r,p,n}$*-modules. Hence,* K *is a splitting field for* $\mathscr{H}_{r,p,n}$*.*
b) *The decomposition matrix of* $\mathscr{H}_{r,p,n}$ *is unitriangular.*

The structure of the Specht modules $S^{\boldsymbol{\lambda}}_t$ and the simple modules $D^{\boldsymbol{\lambda}}_t$ is described in more detail in Theorem 3.48 and Theorem 3.50.

We now have the notation to define the most important l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$.

Definition 1.5. *Suppose that* l *divides* p*,* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$ *and that* $1 \le i \le p_{\boldsymbol{\lambda}}$ *and* $1 \le j \le p_{\boldsymbol{\mu}}$*. The decomposition number* $[S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j]$ *is* l***-splittable*** *if* $p_{\boldsymbol{\lambda}} = l = p_{\boldsymbol{\mu}}$*.*

By the results in section 4, and the general theory developed in [25], the decomposition number $[S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j]$ is p-splittable if and only if $S^{\boldsymbol{\lambda}}_j$ and $D^{\boldsymbol{\mu}}_j$ both have trivial *inertia groups* in the usual sense of Clifford theory.

Now suppose that l divides p and let $m = p/l$. To give an explicit formula for the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ let $V(l)$ be the $l \times l$ Vandermonde matrix

$$V(l) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \varepsilon^{m} & \varepsilon^{2m} & \dots & \varepsilon^{lm} \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon^{(l-1)m} & \varepsilon^{2(l-1)m} & \dots & \varepsilon^{l(l-1)m} \end{pmatrix}.$$

For $1 \le i \le l$ define $V_i(l)$ to be the matrix obtained from $V(l)$ by replacing its ith column with the column vector

$$\begin{pmatrix} d^{l}_{\boldsymbol{\lambda}_m \boldsymbol{\mu}_m} \\ \left(\frac{\mathfrak{g}_{\boldsymbol{\lambda}}}{\mathfrak{g}_{\boldsymbol{\mu}}}\right)^{1} d^{l_1}_{\boldsymbol{\lambda}_m \boldsymbol{\mu}_m} \\ \vdots \\ \left(\frac{\mathfrak{g}_{\boldsymbol{\lambda}}}{\mathfrak{g}_{\boldsymbol{\mu}}}\right)^{l-1} d^{l_{l-1}}_{\boldsymbol{\lambda}_m \boldsymbol{\mu}_m} \end{pmatrix}$$

where $d_{\boldsymbol{\lambda}_m \boldsymbol{\mu}_m} = [S(\boldsymbol{\lambda}^{[1]}, \dots, \boldsymbol{\lambda}^{[m]}) : D(\boldsymbol{\mu}^{[1]}, \dots, \boldsymbol{\mu}^{[m]})]$ and $l_t = \gcd(l, t)$.

Theorem D. *Suppose that K is a field, that $\mathbf{Q}$ is (ε, q)-separated over K and that the decomposition number $[S_i^{\boldsymbol{\lambda}} : D_j^{\boldsymbol{\mu}}]$ is l-splittable, for some l dividing p. Then*

$$[S_i^{\boldsymbol{\lambda}} : D_j^{\boldsymbol{\mu}}] \equiv \frac{\det V_{j-i}(l)}{\det V(l)} \pmod{\operatorname{char} K},$$

for $1 \le i, j \le l = p_{\boldsymbol{\lambda}} = p_{\boldsymbol{\mu}}$. In particular, the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ are known when K is a field of characteristic zero.

A closed formula for $\mathfrak{g}_{\boldsymbol{\lambda}}$ is given in Proposition 3.40 and Remark 3.41, so Theorem D completely determines the splittable decomposition numbers of $\mathscr{H}_{r,p,n}$.

The main idea underpinning Theorem D is the introduction of a new algebra $\mathscr{S}_{r,p,n}$, which is an analogue of the cyclotomic Schur algebra [11] for $\mathscr{H}_{r,p,n}$. We construct Weyl modules and simple modules for $\mathscr{S}_{r,p,n}$ and then compute the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}$ using the **twining characters** of $\mathscr{S}_{r,p,n}$. The twining characters, which generalize of formal characters, compute the trace of certain elements $\vartheta_{\boldsymbol{\lambda}} \in \mathscr{S}_{r,p,n}$ on the weight spaces of some $\mathscr{S}_{r,p,n}$-modules. The map $\vartheta_{\boldsymbol{\lambda}}$ is constructed from the endomorphism $\theta_{\boldsymbol{\lambda}}$ mention earlier so, once again, $\vartheta_{\boldsymbol{\lambda}}$ comes from action of $z_{\mathbf{b}}$ upon certain $\mathscr{S}_{r,p,n}$-modules. Finally, Theorem D is proved using a considerable amount of Clifford theory and some natural functors

$$\bigoplus_{\mathbf{b} \in \mathscr{C}^{\sigma}_{p,n}} \text{Mod-}\mathscr{S}_{r,p,n}(\mathbf{b}) \xrightarrow{\oplus \mathsf{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}} \bigoplus_{\mathbf{b} \in \mathscr{C}^{\sigma}_{p,n}} \text{Mod-}\mathscr{E}_{d,\mathbf{b}} \xrightarrow[\text{Morita}]{\simeq} \text{Mod-}\mathscr{H}_{r,p,n}.$$

where the first functor is an analogue of the Schur functor and the second functor is the restriction of the Morita equivalence of (1.2) to $\mathscr{H}_{r,p,n}$.

Very briefly, the outline of this paper is as follows. Chapter 2 studies the right ideals $V_{\mathbf{b}} = v_{\mathbf{b}} \mathscr{H}_{r,n}$. The main results are Lemma 2.25 which shows the existence of the central element $z_{\mathbf{b}}$, Theorem 2.31 which produces a subalgebra of $\mathscr{H}_{r,n}$ isomorphic to $\mathscr{H}_{d,\mathbf{b}}$, and Theorem 2.35 which is a comparison theorem for the natural trace forms on $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$. In Chapter 3 these results are used to compute the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$, which describe the action of $z_{\mathbf{b}}$ on the Specht modules $S(\boldsymbol{\lambda})$ of $\mathscr{H}_{r,n}$. This proves the first half of Theorem B. Section 3.4 marks the first direct appearance of the algebras $\mathscr{H}_{r,p,n}$. Using seminormal forms we

factorize the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$ in Theorem 3.37, completing the proof of Theorem B. We then use the roots of the scalars $\mathfrak{g}_{\boldsymbol{\lambda}}$ to decompose the Specht modules as $\mathscr{H}_{r,p,n}$-modules, culminating in Theorem 3.48 and Theorem 3.50 which describe the Specht modules and simple modules of $\mathscr{H}_{r,p,n}$, respectively. This completes the proof of Theorem C. Chapter 4 begins by lifting the Morita equivalence (1.2) to a new Morita equivalence between $\mathscr{H}_{r,p,n}$ and a new algebra $\mathscr{E}_d$ in Corollary 4.5. Section 4.2 introduces and studies the algebras $\mathscr{S}_{r,p,n}$, which are analogues of the cyclotomic Schur algebras for $\mathscr{H}_{r,p,n}$. Theorem 4.37 computes the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}$ using its twining characters. Applying the functors mentioned in the last paragraph we then prove Theorem D and hence complete the proof of Theorem A. Finally, in the appendix we prove some technical results whose proofs were deferred from Chapter 2.

Index of notation

$\sim_\sigma$	Equivalence relation $\mathbf{b}\sim\mathbf{b}\langle k\rangle$
$\sim_{\mathbf{b}}$	Equivalence relation $\boldsymbol{\lambda}\sim\boldsymbol{\lambda}\langle k\mathsf{o}_{\mathbf{b}}\rangle$
$\uparrow_B^A, \downarrow_B^A$	Induction & restriction functors
$A(\varepsilon,q,\mathbf{Q})$	$\prod_{i,j,\|k\|<n,1\le t<p}(Q_i-\varepsilon^t q^k Q_j)$
$\mathcal{A}$	$\mathbb{Z}[\dot\varepsilon,\dot q^{\pm1},\dot Q_1^{\pm1},\dots,\dot Q_d^{\pm1},\frac{1}{A(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})}]$
$\mathbf{b}_i^j$	$\delta_{i\le j}(b_i+\cdots+b_j)$
$\mathbf{a}\vee\mathbf{b}$	The concatenation of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{b}\langle z\rangle$	The shift of the sequence $\mathbf{b}$ by z
$\mathscr{C}_{p,n}$	Compositions of n of length p
$\mathrm{ch}\, M$	$\sum_{\boldsymbol\mu}(\dim M_{\boldsymbol\mu})e^{\boldsymbol\mu}$
$\mathrm{ch}_{\mathrm{t}}^{\mathrm{l}}\, M$	$\sum_\gamma \mathrm{Tr}(\vartheta^t_{\boldsymbol\lambda}, M_{\gamma^{l_t}})e^\gamma$
d	r/p
$D(\boldsymbol\lambda)$	Simple module for $\mathscr{H}_{r,n}$
$D_{\mathbf b}(\boldsymbol\lambda)$	Simple module for $\mathscr{H}_{d,\mathbf b}$
$D^{\boldsymbol\lambda}_{i,p}$	Simple module for $\mathscr{E}_{d,\mathbf b}$
$D^{\boldsymbol\lambda}_i$	Simple module for $\mathscr{H}_{r,p,n}$
$\Delta(\boldsymbol\lambda)$	Weyl module for $\mathscr{S}_{r,n}$
$\Delta_{\mathbf b}(\boldsymbol\lambda)$	Weyl module for $\mathscr{S}_{d,\mathbf b}$
$\Delta^{\boldsymbol\lambda}_{i,p}$	Weyl module for $\mathscr{S}_{r,p,n}$
ε	Primitive pth root of unity in K
$\dot\varepsilon$	Primitive pth root of unity in $\mathbb{C}$
$e_{\mathbf b}$	The idempotent $z_{\mathbf b}^{-1}\cdot v_{\mathbf b}T_{\mathbf b}$
$\mathscr{E}_{d,\mathbf b}$	$=\mathrm{End}_{\mathscr{H}_{r,p,n}}(V_{\mathbf b})$
$\mathcal{F}$	The field of fractions of $\mathcal{A}$
$\mathfrak f_{\boldsymbol\lambda}$	The scalar: $z_{\mathbf b}\downarrow_{S(\boldsymbol\lambda)}=\mathfrak f_{\boldsymbol\lambda}\,\mathrm{id}_{S(\boldsymbol\lambda)}$
$\mathfrak f^{(t)}_{\boldsymbol\lambda}$	$=\mathfrak f^{(t:m)}_{\boldsymbol\lambda}$, a factor of $\mathfrak f_{\boldsymbol\lambda}$
$\mathfrak g_{\boldsymbol\lambda}$	$\mathfrak f_{\boldsymbol\lambda}=\varepsilon^{\frac12 d\mathsf{o}_{\boldsymbol\lambda}n(1-p_{\boldsymbol\lambda})}\mathfrak g_{\boldsymbol\lambda}^{p_{\boldsymbol\lambda}}$
$\mathtt{H}_{\mathbf b}$	A functor Mod-$\mathscr{H}_{d,\mathbf b}\to$ Mod-$\mathscr{H}_{r,n}$
θ'_t	The map $h\mapsto Y_t h$
$\theta'_{t,m}$	The map $h\mapsto Y_{t,m}h$
$\theta_{t,m}$	$=\sigma^m\circ\theta'_{t,m}$
$\theta_{\mathbf b}$	$=\theta_{0,\mathsf{o}_p(\mathbf b)}$ restricted to $M^{\boldsymbol\lambda}_{\mathbf b}$
$\theta_{\boldsymbol\lambda}$	$=\theta_{0,\mathsf{o}_p(\boldsymbol\lambda)}$ restricted to $S(\boldsymbol\lambda)$
$\vartheta_{\mathbf b}$	$\theta_{\mathbf b}$ restricted to $M^{\boldsymbol\lambda}_{\mathbf b}$
$\vartheta_{\boldsymbol\lambda}$	$\vartheta_{\mathbf b}^{p_{\mathbf b/\boldsymbol\lambda}}$
$\widehat\Theta_{\mathbf b},\Theta_{\mathbf b}$	Two linear maps $\mathscr{H}_{d,\mathbf b}\to\mathscr{H}_{r,n}$
$\mathscr{H}^R_{r,n}$	Hecke algebra of type $G(r,1,n)$
$\mathscr{H}^R_{r,p,n}$	Hecke algebra of type $G(r,p,n)$
$\mathscr{H}_{d,\mathbf b}$	$=\mathscr{H}_{d,b_1}(\varepsilon\mathbf Q)\otimes\cdots\otimes\mathscr{H}_{d,b_p}(\varepsilon^p\mathbf Q)$
$\widehat{\mathscr{H}_{d,\mathbf b}}$	$=\mathscr{H}_{d,\mathbf b}\cdot e_{\mathbf b}\cong\mathscr{H}_{d,\mathbf b}$
$\mathscr{K}_{r,n}$	Kleshchev multipartitions in $\mathscr{P}_{r,n}$
$\mathscr{K}_{d,\mathbf b}$	Kleshchev multipartitions in $\mathscr{P}_{d,\mathbf b}$
$L(\boldsymbol\lambda)$	Simple module for $\mathscr{S}_{r,n}$
$L_{\mathbf b}(\boldsymbol\lambda)$	Simple module for $\mathscr{S}_{d,\mathbf b}$
$L^{\boldsymbol\lambda}_{i,p}$	Simple module for $\mathscr{S}_{r,p,n}$
$M^{\boldsymbol\lambda}_{\mathbf b}$	Permutation module in $V_{\mathbf b}$
$\mathsf{o}_p(\mathbf b)$	$=\min\{\,z>0\mid \mathbf b\langle z\rangle=\mathbf b\,\}$
$\mathsf{o}_{\mathbf b}$	$=\mathsf{o}_p(\mathbf b)$
$\mathsf{o}_{\boldsymbol\lambda}$	$=\mathsf{o}_p(\boldsymbol\lambda)$
$p_{\boldsymbol\lambda}$	$p/\mathsf{o}_p(\boldsymbol\lambda)=p/\mathsf{o}_{\boldsymbol\lambda}$
$p_{\mathbf b/\boldsymbol\lambda}$	$p_{\mathbf b}/p_{\boldsymbol\lambda}=\mathsf{o}_{\boldsymbol\lambda}/\mathsf{o}_{\mathbf b}$
$p_{\boldsymbol\mu/\boldsymbol\lambda}$	$p_{\boldsymbol\mu}/p_{\boldsymbol\lambda}=\mathsf{o}_{\boldsymbol\lambda}/\mathsf{o}_{\boldsymbol\mu}$
$\mathscr{P}_{r,n}$	The set of r–multipartitions of n
$\mathscr{P}_{d,\mathbf b}$	$\{\,\boldsymbol\lambda\in\mathscr{P}_{r,n}\mid b_i=\|\boldsymbol\lambda^{[i]}\|,1\le i\le d\,\}$
$\boldsymbol\lambda^{[i]}$	$(\lambda^{(d(i-1)+1)},\dots,\lambda^{(di)})$
$\mathcal{L}^{(s)}_k$	$\prod_{i=1}^d(L_k-\varepsilon^sQ_i)$
$\mathcal{L}^{(i,j)}_{l,m}$	$\prod_{l\le k\le m}\prod_{s\in I_{ij}}\mathcal{L}^{(s)}_k$
$\mathbf Q$	$(Q_1,\dots,Q_d)$
$\mathbf Q^{\vee\varepsilon}$	$\varepsilon\mathbf Q\vee\varepsilon^2\mathbf Q\vee\cdots\vee\varepsilon^p\mathbf Q$
σ	$\mathscr{H}_{r,n}\to\mathscr{H}_{r,n}; T_i\mapsto\varepsilon^{\delta_{i0}}T_i$
$\hat\sigma$	An automorphism of $\mathscr{S}_{r,p,n}$
s_i	$(i,i+1)\in\mathfrak{S}_n$
$S(\boldsymbol\lambda)$	Specht module for $\mathscr{H}_{r,n}$
$S_{\mathbf b}(\boldsymbol\lambda)$	Specht module for $\mathscr{H}_{d,\mathbf b}$
$S^{\boldsymbol\lambda}_{i,p}$	Specht module for $\mathscr{E}_{d,\mathbf b}$
$S^{\boldsymbol\lambda}_t$	Specht module for $\mathscr{H}_{r,p,n}$
$\dot{\mathfrak s}_{\boldsymbol\lambda}$	Schur element of $S(\boldsymbol\lambda)$
$\dot{\mathfrak s}^{\mathbf b}_{\boldsymbol\lambda}$	Schur element of $S_{\mathbf b}(\boldsymbol\lambda)$
$\mathscr{S}_{r,n}$	Cyclotomic Schur algebra for $\mathscr{H}_{r,n}$
$\mathscr{S}_{d,\mathbf b}$	Cyclotomic Schur algebra for $\mathscr{H}_{d,\mathbf b}$
$\mathscr{S}_{r,p,n}$	Cyclotomic Schur algebra for $\mathscr{H}_{r,p,n}$

$\operatorname{Std}(\boldsymbol{\lambda})$	Standard $\boldsymbol{\lambda}$-tableaux	$V_{\mathbf{b}}^{(t)}$	$= v_{\mathbf{b}}^{(t)}\mathscr{H}_{r,n}$
$T_{a,b}$	$T_{w_{a,b}}$	$\{v_{\mathfrak{s}}^{(tm)}\}$	Seminormal basis of $S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$
$T_{a,b}^{\langle k\rangle}$	T_w, where $w = w_{a,b}^{\langle k\rangle}$	$w_{a,b}^{\langle k\rangle}$	$(s_{a+b+k-1}\dots s_{k+1})^b$
τ	$h \mapsto T_0^{-1}hT_0$	$w_{\mathbf{b}}$	$w_{b_{p-1},\mathbf{b}_p^p}^{\langle \mathbf{b}_1^{p-2}\rangle}\dots w_{b_2,\mathbf{b}_3^p}^{\langle \mathbf{b}_1^1\rangle} w_{b_1,\mathbf{b}_2^p}$
$v_{\mathbf{b}}$	$v_{\mathbf{b}}(\mathbf{Q}) = v_{\mathbf{b}}^+u_{\mathbf{b}}^+ = u_{\mathbf{b}}^-v_{\mathbf{b}}^-$	Y_t	$\mathcal{L}_{1,b_t}^{(t+1,t+p-1)}T_{b_t,n-b_t}$
$v_{\mathbf{b}}^{(t)}$	$v_{\mathbf{b}}(\varepsilon^t\mathbf{Q})$	$Y_{t,m}$	$Y_{tm}Y_{tm-1}\dots Y_{t(m-1)+1}$
$V_{\mathbf{b}}$	The ideal $v_{\mathbf{b}}\mathscr{H}_{r,n}$	$z_{\mathbf{b}}$	Central element of $\mathscr{H}_{d,\mathbf{b}}$

2. Hecke algebras of type $G(r,1,n)$ and the central elements $z_{\mathbf{b}}$

The main objects studied in this paper are the Hecke algebras of type $G(r,p,n)$, where $r = pd$ for some $d \in \mathbb{N}$. These algebras are deformations of the group rings of the corresponding complex reflection groups of type $G(r,p,n)$.

The complex reflection groups of type $G(r,1,n)$ are the groups $(Z/r\mathbb{Z}) \wr \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on $\{1,2,\dots,n\}$. The group of type $G(pd,p,n)$ is a normal subgroup of $(\mathbb{Z}/pd\mathbb{Z}) \wr \mathfrak{S}_n$ of index p which is fixed by an automorphism of $G(pd,1,n)$. Similarly, if $n \geq 3$ then the Hecke algebra of type $G(pd,p,n)$ can be defined as the fixed point subalgebra of an automorphism of the Hecke algebra of type $G(pd,1,n)$.

In this chapter we define the Hecke algebras of types $G(r,1,n)$ and $G(r,p,n)$ and begin to set up the machinery that we need in order to prove Theorem B. The highlights of this chapter are Lemma 2.25, which proves the existence of the central elements $z_{\mathbf{b}}$ from Theorem B, and Theorems 2.31 and 2.35 which, in Chapter 3, will allow us to compute the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$ and $\mathfrak{g}_{\boldsymbol{\lambda}}$ in Theorem B.

2.1. **Cyclotomic Hecke algebras.** We begin by defining the Hecke algebras $\mathscr{H}_{r,n}$ and $\mathscr{H}_{r,p,n}$ from the introduction and recalling the machinery that we need from [25].

Throughout this paper we fix positive integers n, r, p and d such that $n \geq 3$, $p > 1$ and $r = pd$. Let R be a commutative ring which contains a primitive pth root of unity ε.

Suppose that $q, Q_1, \dots, Q_r$ are invertible elements of K. The **Ariki-Koike algebra** $\mathscr{H}_{r,n}^R(q, Q_1, \dots, Q_r)$ is the unital associative R-algebra with generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned}
(T_0 - Q_1)\dots(T_0 - Q_r) &= 0, \\
(T_i - q)(T_i + 1) &= 0, && \text{for } 1 \leq i \leq n-1, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\
T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i, && \text{for } 1 \leq i \leq n-2, \\
T_iT_j &= T_jT_i, && \text{for } 0 \leq i < j-1 \leq n-2.
\end{aligned}$$

To define the Hecke algebras of type $G(r,p,n)$ we fix $\mathbf{Q} = (Q_1, \dots, Q_d) \in K^d$ and replace the $(Q_1, \dots, Q_r)$ in the above definition by $\mathbf{Q}^{\vee\boldsymbol{\varepsilon}}$, where

$$\mathbf{Q}^{\vee\boldsymbol{\varepsilon}} = \varepsilon\mathbf{Q} \vee \varepsilon^2\mathbf{Q} \vee \dots \vee \varepsilon^p\mathbf{Q} = (\varepsilon Q_1, \dots, \varepsilon Q_d, \dots, \varepsilon^p Q_1, \dots, \varepsilon^p Q_d). \tag{2.1}$$

We set $\mathscr{H}_{r,n}^R(\mathbf{Q}^{\vee\boldsymbol{\varepsilon}}) := \mathscr{H}_{r,n}^R(q, \mathbf{Q}^{\vee\boldsymbol{\varepsilon}})$. Then the relation for T_0 in $\mathscr{H}_{r,n}(\mathbf{Q}^{\vee\boldsymbol{\varepsilon}})$ can be written as $(T_0^p - Q_1^p)\dots(T_0^p - Q_d^p) = 0$. When R and $q, Q_1, \dots, Q_d$ are understood we write $\mathscr{H}_{r,n} = \mathscr{H}_{r,n}(\mathbf{Q}^{\vee\boldsymbol{\varepsilon}})$.

Definition 2.2. *The **cyclotomic Hecke algebra of type** $G(r,p,n)$ is the subalgebra $\mathscr{H}_{r,p,n} = \mathscr{H}_{r,p,n}(\mathbf{Q})$ of $\mathscr{H}_{r,n}(\mathbf{Q}^{\vee\boldsymbol{\varepsilon}})$ generated by the elements $T_0^p, T_u = T_0^{-1}T_1T_0$ and $T_1, T_2, \cdots, T_{n-1}$.*

In this paper we are interested in understanding the decomposition matrices of the algebras $\mathscr{H}_{r,p,n}$. Although this will not be apparent for quite some time, we have chosen the ordering of the 'cyclotomic parameters' $\varepsilon\mathbf{Q} \vee \varepsilon^2\mathbf{Q} \vee \cdots \vee \varepsilon^p\mathbf{Q}$ in order to ensure that the labelling of the irreducible modules for $\mathscr{H}_{r,n}$ and $\mathscr{H}_{r,p,n}$ are compatible in the sense of Theorem C.

The algebra $\mathscr{H}_{r,n}$ comes equipped with two automorphisms σ and τ which are useful when studying $\mathscr{H}_{r,p,n}$. Let σ be the unique automorphism of $\mathscr{H}_{r,n}$ such that

$$\sigma(T_0) = \varepsilon T_0 \quad \text{and} \quad \sigma(T_i) = T_i, \qquad \text{for } 1 \le i < n, \tag{2.3}$$

and define τ by $\tau(h) = T_0^{-1} h T_0$, for $h \in \mathscr{H}_{r,n}$. It is straightforward to check that $\mathscr{H}_{r,p,n} = \{\, h \in \mathscr{H}_{r,n} \mid \sigma(h) = h \,\}$ is the set of σ-fixed points in $\mathscr{H}_{r,n}$ and that τ restricts to an automorphism of $\mathscr{H}_{r,p,n}$. Moreover, σ is an automorphism of $\mathscr{H}_{r,n}$ of order p and τ is an automorphism of $\mathscr{H}_{r,p,n}$ with the property that τ^p is an inner automorphism of $\mathscr{H}_{r,p,n}$.

Fix a modular system $(F, \mathcal{O}, K)$ "with parameters" for $\mathscr{H}_{r,p,n}$. That is, fix an algebraically closed field F of characteristic zero, a discrete valuation ring $\mathcal{O}$ with maximal ideal π and with residue field $K \cong \mathcal{O}/\pi$, together with parameters $\hat{q}, \hat{Q}_1, \dots, \hat{Q}_d \in \mathcal{O}^\times$ such that $q = \hat{q} + \pi$ and $Q_i = \hat{Q}_i + \pi$ for each i. Let $\mathscr{H}^F_{r,p,n} = \mathscr{H}^F_{r,p,n}(\hat{\mathbf{Q}})$ be the Hecke algebra of type $G(r,p,n)$ over F with parameters $\hat{q}$ and $\hat{\mathbf{Q}} = (\hat{Q}_1, \dots, \hat{Q}_d)$ and similarly let $\mathscr{H}^{\mathcal{O}}_{r,p,n} = \mathscr{H}_{r,p,n}(\hat{\mathbf{Q}})$ and write $\mathscr{H}^K_{r,p,n} = \mathscr{H}_{r,p,n}(\mathbf{Q})$. We assume that $\mathscr{H}^F_{r,p,n}$ is semisimple. By Lemma 2.7 below, $\mathscr{H}^F_{r,p,n} \cong \mathscr{H}^{\mathcal{O}}_{r,p,n} \otimes_{\mathcal{O}} F$ and $\mathscr{H}^K_{r,p,n} \cong \mathscr{H}^{\mathcal{O}}_{r,p,n} \otimes_{\mathcal{O}} K$. Hence, by choosing $\mathcal{O}$–lattices we can talk of modular reduction from $\mathscr{H}^F_{r,p,n}$–Mod to $\mathscr{H}^K_{r,p,n}$–Mod.

The automorphisms σ and τ commute with modular reduction. Hence, we have compatible automorphisms σ and τ on $\mathscr{H}^F_{r,n}$ and on $\mathscr{H}^K_{r,n}$.

Let $R \in \{F, K\}$ and let M be an $\mathscr{H}^R_{r,p,n}$–module. Then we define a new $\mathscr{H}^R_{r,p,n}$–module M^τ by "twisting" the action of $\mathscr{H}^R_{r,p,n}$ using the automorphism τ. Explicitly, $M^\tau = M$ as a vector space and the $\mathscr{H}^R_{r,p,n}$–action on M^τ is defined by

$$m \cdot h = m\tau(h), \qquad \text{for all } m \in M \text{ and } h \in \mathscr{H}^R_{r,p,n}.$$

If M is an $\mathscr{H}_{r,p,n}$-module then $M \cong M^{\tau^p}$ because τ^p is an inner automorphism of $\mathscr{H}_{r,p,n}$. Therefore, there is a natural action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ on the set of isomorphism classes of $\mathscr{H}^R_{r,p,n}$–modules. The **inertia group** of M is the group

$$I_M = \{\, k \mid 0 \le k < p, M \cong M^{\tau^k} \,\},$$

which we consider as a subgroup of $\mathbb{Z}/p\mathbb{Z}$.

Suppose that S is an irreducible $\mathscr{H}^F_{r,p,n}$-module and let S_K be $\mathscr{H}^K_{r,p,n}$-module obtained from S by modular reduction. Let D be an irreducible $\mathscr{H}^K_{r,p,n}$-module. By [25, Corollary 5.6], the decomposition number $[S_K : D]$ is p**-splittable** in the sense of Definition 2.2 if and only if $I_S = \{0\} = I_D$.

If $\alpha \in K$ then the q**-orbit** of α is the set $\{\, q^b\alpha \mid b \in \mathbb{Z} \,\}$. Similarly, the (ε, q)**-orbit** of α is$\{\, \varepsilon^a q^b \alpha \mid a, b \in \mathbb{Z} \,\}$.

One of the main results of [25] is the following.

Theorem 2.4 (([25, Theorem B]))**.** *The decomposition numbers of the cyclotomic Hecke algebras of type $G(r,p,n)$ are completely determined by the l-splittable decomposition numbers of certain cyclotomic Hecke algebras $\mathscr{H}_{s,l,m}(\mathbf{Q}')$, where l*

divides p, $1 \le s \le r$, $1 \le m \le n$ and where the parameters $\mathbf{Q}'$ are contained in a single (ε, q)-orbit.

Hence, to prove Theorem A it is enough to compute all of the p-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ and to show that they are determined by the decomposition numbers of Hecke algebras of type $G(s,1,m)$, where $s \ge 1$ divides r and $1 \le m \le n$.

To compute the p-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ we make extensive use of the following result which is a more precise statement of (1.2).

Theorem 2.5 (([12, 25])). *Suppose that $(Q_1,\dots,Q_r) = \mathbf{Q}_1 \vee \cdots \vee \mathbf{Q}_\gamma$, where $Q_i \in \mathbf{Q}_\alpha$ and $Q_j \in \mathbf{Q}_\beta$ are in the same q–orbit only if $\alpha = \beta$. Let $d_\alpha = |\mathbf{Q}_\alpha|$, for $1 \le \alpha \le \gamma$. Then $\mathscr{H}_{r,n}(q,\mathbf{Q})$ is Morita equivalent to the algebra*

$$\bigoplus_{b_1+\cdots+b_\gamma=n} \mathscr{H}_{d_1,b_1}(q,\mathbf{Q}_1)\otimes\cdots\otimes\mathscr{H}_{d_\gamma,b_\gamma}(q,\mathbf{Q}_\gamma).$$

Recall from (1.1) that $\mathbf{Q}$ is (ε,q)-separated if

$$\prod_{1\le i,j\le d}\ \prod_{-n<k<n}\ \prod_{1\le t<p} \big(Q_i - \varepsilon^t q^k Q_j\big) \ne 0.$$

To prove our main results we apply Theorem 2.5 to the decomposition $\mathbf{Q}^{\vee\boldsymbol{\varepsilon}} = \varepsilon\mathbf{Q} \vee \varepsilon^2\mathbf{Q} \vee\cdots\vee \varepsilon^p\mathbf{Q}$. This will allow us to understand the Specht modules $S(\boldsymbol{\lambda})$ of $\mathscr{H}_{r,n}$ in terms of the Specht modules of the algebras $\mathscr{H}_{d,\mathbf{b}} = \mathscr{H}_{d,b_1}(\varepsilon\mathbf{Q})\otimes\cdots\otimes\mathscr{H}_{d,b_p}(\varepsilon^p\mathbf{Q})$. The main benefit in doing this is that because of our ordering on the parameters in $\mathbf{Q}^{\vee\boldsymbol{\varepsilon}}$ it is easier to understand the action of the automorphisms σ on $\mathscr{H}_{d,\mathbf{b}}$-modules and this gives us a way to understand the action of σ on $S(\boldsymbol{\lambda})$. In turn, this will allow to decompose the restriction of $S(\boldsymbol{\lambda})$ to $\mathscr{H}_{r,p,n}$.

When using Clifford theory to understand the representation theory of the group $G(r,p,n)$ in terms of the representation theory of $G(r,1,n)$ it is natural to assume that the characteristic of the field does not divide p, which is the index of $G(r,p,n)$ in $G(r,1,n)$. As the following results indicates, (ε,q)-separation can be viewed as a quantum analogue of this condition.

Lemma 2.6. *Suppose that $\mathbf{Q} \in K^d$ and $R = K$ is a field. Then:*

a) *Suppose that $\mathbf{Q}$ is (ε,q)-separated over K. Then*

$$\prod_{-n<k<n}\ \prod_{1\le t<p} (1-\varepsilon^t q^k) \ne 0.$$

In particular, $\langle\varepsilon\rangle\cap\langle q\rangle = \{1\}$ if $q^k = 1$ for some $1 \le k < n$.

b) *Suppose that $\langle\varepsilon\rangle\cap\langle q\rangle = \{1\}$ and that $\mathbf{Q}$ is contained in a single q-orbit. Then $\mathbf{Q}$ is (ε,q)-separated over K.*

Proof. Part (a) follows by looking at the terms in the product corresponding to $i = j$ in (1.1). For (b) observe that if $\langle\varepsilon\rangle\cap\langle q\rangle = \{1\}$ then $\varepsilon^t q^k \ne 1$ if $t \ne p$. Since $\mathbf{Q}$ is contained in a single q-orbit this implies the result. □

2.2. **Jucys-Murphy elements and a basis for $\mathscr{H}_{r,n}$.** In order to define a basis for $\mathscr{H}_{r,n}$ let $\mathfrak{S}_n$ be the symmetric group on n letters and let $s_i = (i,i+1) \in \mathfrak{S}_n$ be a simple transposition, for $1 \le i < n$. Then $\{s_1,\dots,s_{n-1}\}$ are the standard Coxeter generators of the symmetric group $\mathfrak{S}_n$. Let $\ell\colon\mathfrak{S}_n \longrightarrow \mathbb{N}$ be the length function on $\mathfrak{S}_n$, so that $\ell(w) = k$ if k is minimal such that $w = s_{i_1}\dots s_{i_k}$, where

$1 \le i_1, \dots, i_k < n$. As the type A braid relations hold in $\mathscr{H}_{r,n}$ for each $w \in \mathfrak{S}_n$ there is a well–defined element $T_w \in \mathscr{H}_{r,n}$, where $T_w = T_{i_1} \dots T_{i_k}$ whenever $w = s_{i_1} \dots s_{i_k}$ and $k = \ell(w)$.

Set $L_1 = T_0$ and $L_{k+1} = q^{-1} T_k L_k T_k$, for $k = 1, \dots, n-1$. These elements L_i are the Jucys–Murphy elements of $\mathscr{H}_{r,n}$ and they generate a commutative subalgebra of $\mathscr{H}_{r,n}$.

Lemma 2.7. a) [4, Theorem 3.10] *The algebra $\mathscr{H}_{r,n}$ is free as an R-module with basis $\{ L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \le a_j < r$ and $w \in \mathfrak{S}_n \}$.*

b) [2, Proposition 1.6] *The algebra $\mathscr{H}_{r,p,n}$ is free as an R-module with basis $\{ L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \le a_j < r, a_1 + \dots + a_n \equiv 0 \pmod p$ and $w \in \mathfrak{S}_n \}$.*

Inspecting the relations, there is a unique anti-isomorphism $*$ of $\mathscr{H}_{r,n}$ which fixes each of the generators $T_0, T_1, \dots, T_{n-1}$ of $\mathscr{H}_{r,n}$. We have $T_w^* = T_{w^{-1}}$ and $L_k^* = L_k$, for $1 \le k \le n$.

We will use the following well-known properties of the Jucys-Murphy elements without mention.

Lemma 2.8 ((cf. [4, Lemma 3.3]))**.** *Suppose that $1 \le i < n$ and $1 \le k \le n$. Then*

a) *T_i and L_k commute if $i \ne k, k-1$.*
b) *T_k commutes with $L_k L_{k+1}$ and $L_k + L_{k+1}$.*
c) *$T_k L_k = L_{k+1}(T_k - q + 1)$ and $T_k L_{k+1} = L_k T_k + (q-1) L_{k+1}$.*

For integers k and s, with $1 \le k \le n$ and $1 \le s \le p$, set

$$\mathcal{L}_k^{(s)} = \prod_{i=1}^{d} (L_k - \varepsilon^s Q_i).$$

More generally, if $1 \le l \le m \le n$ and $1 \le i, j \le p$ then set

$$\mathcal{L}_{l,m}^{(i,j)} = \prod_{\substack{l \le k \le m \\ s \in I_{ij}}} \mathcal{L}_k^{(s)} = \prod_{\substack{l \le k \le m \\ s \in I_{ij}}} \prod_{t=1}^{d} (L_k - \varepsilon^s Q_t),$$

where $I_{ij} = \{i, i+1, \dots, j\}$, if $i \le j$, and $I_{ij} = \{1, 2 \dots, j, i, i+1, \dots, p\}$ if $i > j$.

A key property of the Jucys-Murphy elements of $\mathscr{H}_{r,n}$ is that T_i commutes with any polynomial in $L_1, \dots, L_n$ which is symmetric with respect to L_i and L_{i+1}. In particular, any symmetric polynomial in $L_1, \dots, L_n$ is central in $\mathscr{H}_{r,n}$. Hence, we have the following.

Lemma 2.9. *Suppose that $1 \le l < m \le n$ and $1 \le t \le p$. Then*

$$T_i \mathcal{L}_{l,m}^{(t)} = \mathcal{L}_{l,m}^{(t)} T_i \qquad \textit{and} \qquad L_j \mathcal{L}_{l,m}^{(t)} = \mathcal{L}_{l,m}^{(t)} L_j,$$

for all i, j such that $1 \le i < n$, $1 \le j \le n$ and $i \ne l-1, m$.

Throughout this paper we will need some special permutations. For non-negative integers a, b with $0 < a + b \le n$ we set $w_{a,b} = (s_{a+b-1} \dots s_1)^b$. (In particular, $w_{a,0} = 1 = w_{0,b}$.) If we write $w_{a,b} \in \mathfrak{S}_{a+b}$ as a permutation in two-line notation then

$$w_{a,b} = \begin{pmatrix} 1 & \cdots & a & a+1 & \cdots & a+b \\ b+1 & \cdots & a+b & 1 & \cdots & b \end{pmatrix}. \tag{2.10}$$

For simplicity, we write $T_{a,b} = T_{w_{a,b}}$. Similarly, if k is a non-negative integer such that $0 < a+b+k \le n$ then we set $w_{a,b}^{\langle k\rangle} = (s_{a+b+k-1} \dots s_{k+1})^b$. Then $w_{a,b} = w_{a,b}^{\langle 0\rangle}$ and, abusing notation slightly, we write $T_{a,b}^{\langle k\rangle} = T_{w_{a,b}^{\langle k\rangle}}$.

The following result is easily checked.

Lemma 2.11. *Suppose that a, b and c are non-negative integers such that $a+b+c \le n$. Then $w_{a,b+c} = w_{a,b} w_{a,c}^{\langle b\rangle}$ and $w_{a+b,c} = w_{b,c}^{\langle a\rangle} w_{a,c}$, with the lengths adding. Consequently, $T_{a,b+c} = T_{a,b} T_{a,c}^{\langle b\rangle}$ and $T_{a+b,c} = T_{b,c}^{\langle a\rangle} T_{a,c}$. Moreover, $T_i T_{a,b}^{\langle c\rangle} = T_{a,b}^{\langle c\rangle} T_{(i) w_{a,b}^{\langle c\rangle}}$ if $1 \le i < n$ and $i \ne a+c$.*

2.3. **The elements $v_{\mathbf{b}}$ and $v_{\mathbf{b}}^{(t)}$.** As remarked in the introduction, all of the results in this paper rely on the Morita equivalence of Theorem 2.5. This equivalence is induced by certain $(\mathscr{H}_{d,\mathbf{b}}, \mathscr{H}_{r,n})$-bimodules $V_{\mathbf{b}} = v_{\mathbf{b}} \mathscr{H}_{r,n}$. In this section we define these modules and, in Proposition 2.13, give one of the key properties of the elements $v_{\mathbf{b}}$.

Recall from the introduction that $\mathscr{C}_{p,n}$ is the set of compositions of n into p parts. Thus, $\mathbf{b} \in \mathscr{C}_{p,n}$ if and only if $\mathbf{b} = (b_1, \dots, b_p)$, $b_1 + \dots + b_p = n$ and $b_i \ge 0$, for all i. Finally, if $\mathbf{b} \in \mathscr{C}_{p,n}$ and i and j are integers then we set $\mathbf{b}_i^j = b_i + \dots + b_j$ if $i \le j$ and $b_i^j = 0$ if $i > j$.

The following elements of $\mathscr{H}_{r,n}$ were introduced in [25, Definition 2.4]. They play an important role throughout this paper.

Definition 2.12. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$. Let*

$$v_{\mathbf{b}}(\mathbf{Q}) = \mathcal{L}_{1,b_p}^{(1,p-1)} T_{b_p,\mathbf{b}_1^{p-1}} \mathcal{L}_{1,b_{p-1}}^{(1,p-2)} T_{b_{p-1},\mathbf{b}_1^{p-2}} \dots \mathcal{L}_{1,b_2}^{(1,1)} T_{b_2,\mathbf{b}_1^1} \mathcal{L}_{1,\mathbf{b}_1^1}^{(2)} \mathcal{L}_{1,\mathbf{b}_1^2}^{(3)} \dots \mathcal{L}_{1,\mathbf{b}_1^{p-1}}^{(p)}$$

We write $v_{\mathbf{b}} = v_{\mathbf{b}}(\mathbf{Q})$ and for $t \in \mathbb{Z}$ set $v_{\mathbf{b}}^{(t)} = v_{\mathbf{b}}(\varepsilon^t \mathbf{Q})$.

Set $V_{\mathbf{b}} = v_{\mathbf{b}} \mathscr{H}_{r,n}$ and, more generally, let $V_{\mathbf{b}}^{(t)} = v_{\mathbf{b}}^{(t)} \mathscr{H}_{r,n}$.

The element $v_{\mathbf{b}}$ can be written in many different (and useful) ways. The proof of the next result requires several long and uninspiring calculations, so we refer the reader to Proposition A3 in the appendix for the proof.

Proposition 2.13. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $1 \le j \le p$. Then*

$$v_{\mathbf{b}} = \prod_{j \le k < p} \mathcal{L}_{1,b_{k+1}}^{(j,k)} T_{b_{k+1},\mathbf{b}_j^k} \cdot \prod_{1 \le i < j} \mathcal{L}_{1,\mathbf{b}_{i+1}^p}^{(i)} \cdot \prod_{j < k \le p} \mathcal{L}_{1,\mathbf{b}_j^{k-1}}^{(k)} \cdot \prod_{1 < i \le j} T_{\mathbf{b}_i^p, b_{i-1}} \mathcal{L}_{1,b_{i-1}}^{(i,p)},$$

where all products are read from left to right with <u>decreasing</u> values of i and k.

Corollary 2.14. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $t \in \mathbb{Z}$. Then $v_{\mathbf{b}}^{(t)} \in \mathcal{L}_{1,\mathbf{b}_2^p}^{(t+1)} \mathscr{H}_{r,n}$.*

Proof. It is enough to consider the case when $t = 0$ and $v_{\mathbf{b}}^{(t)} = v_{\mathbf{b}}$. In this case the result follows by taking $j = p$ in Proposition 2.13. □

For $t = 1, \dots, p$, let $Y_t = \mathcal{L}_{1,b_t}^{(t+1,t+p-1)} T_{b_t,n-b_t}$. If $\mathbf{a} = (a_1, a_2, \dots, a_m)$ is any sequence then set $\mathbf{a}\langle k\rangle = (a_{k+1}, a_{k+2}, \dots, a_{k+m})$, where $a_{i+jm} := a_i$ for $j \in \mathbb{Z}$.

Corollary 2.15. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $1 \le t \le p$. Then*

$$Y_t v_{\mathbf{b}\langle t-1\rangle}^{(t-1)} = v_{\mathbf{b}\langle t\rangle}^{(t)} Y_t^*.$$

Proof. It is enough to consider the case $t = 1$. Taking $j = 2$ in Proposition 2.13,

$$\begin{aligned} Y_t v_{\mathbf{b}} &= \mathcal{L}_{1,b_1}^{(2,p)} T_{b_1,\mathbf{b}_2^p} \cdot \mathcal{L}_{1,b_p}^{(2,p-1)} T_{b_p,\mathbf{b}_2^{p-1}} \dots \mathcal{L}_{1,b_3}^{(2,2)} T_{b_3,\mathbf{b}_2^2} \mathcal{L}_{1,\mathbf{b}_2^2}^{(3)} \dots \mathcal{L}_{1,\mathbf{b}_2^{p-1}}^{(p)} \\ &\qquad \times \mathcal{L}_{1,\mathbf{b}_2^p}^{(1)} T_{\mathbf{b}_2^p,b_1} \mathcal{L}_{1,b_1}^{(2,p)} \\ &= v_{\mathbf{b}\langle 1\rangle}^{(1)} T_{\mathbf{b}_2^p,b_1} \mathcal{L}_{1,b_1}^{(2,p)}, \end{aligned}$$

as required. □

The point of Corollary 2.15 is that left multiplication by Y_t defines an $\mathscr{H}_{r,n}$-module homomorphism from $V_{\mathbf{b}\langle t-1\rangle}^{(t-1)} = v_{\mathbf{b}\langle t-1\rangle}^{(t-1)} \mathscr{H}_{r,n}$ to $V_{\mathbf{b}\langle t\rangle}^{(t)} = v_{\mathbf{b}\langle t\rangle}^{(t)} \mathscr{H}_{r,n}$

Definition 2.16. *Suppose that $1 \le t \le p$ and $\mathbf{b} \in \mathscr{C}_{p,n}$. Then θ'_t is the $\mathscr{H}_{r,n}$-module homomorphism*

$$\theta'_t : V_{\mathbf{b}\langle t-1\rangle}^{(t-1)} \longrightarrow V_{\mathbf{b}\langle t\rangle}^{(t)}; x \mapsto Y_t x,$$

for all $x \in V_{\mathbf{b}\langle t-1\rangle}^{(t-1)}$.

Since $v_{\mathbf{b}} = v_{\mathbf{b}\langle p\rangle}^{(p)}$, composing the maps $\theta'_p \circ \dots \circ \theta'_1$ gives an $\mathscr{H}_{r,n}$-module endomorphism of $v_{\mathbf{b}} \mathscr{H}_{r,n}$. We need another description of this map.

Proposition 2.17. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$. Then $Y_p Y_{p-1} \dots Y_2 Y_1 = v_{\mathbf{b}} T_{\mathbf{b}}$.*

This result is proved in the appendix as Proposition A4.

2.4. **The central element $z_{\mathbf{b}}$.** The aim of this section is to prove the existence of the central element $z_{\mathbf{b}}$ which appears in Theorem B. We start by studying the elements $Y_t v_{\mathbf{b}\langle t-1\rangle}^{(t-1)}$. Generalising (2.10), for $\mathbf{b} \in \mathscr{C}_{p,n}$ set

$$w_{\mathbf{b}} = w_{b_{p-1},\mathbf{b}_p^p}^{\langle \mathbf{b}_1^{p-2}\rangle} w_{b_{p-2},\mathbf{b}_{p-1}^p}^{\langle \mathbf{b}_1^{p-3}\rangle} \dots w_{b_2,\mathbf{b}_3^p}^{\langle \mathbf{b}_1^1\rangle} w_{b_1,\mathbf{b}_2^p}.$$

In two-line notation, $w_{\mathbf{b}}$ is the permutation

$$\begin{pmatrix} 1 & \dots & \mathbf{b}_1^1 & \mathbf{b}_1^1+1 & \dots & \mathbf{b}_1^2 & \mathbf{b}_1^2+1 & \dots & \mathbf{b}_1^{p-1}+1 & \dots & \mathbf{b}_1^p \\ \mathbf{b}_2^p+1 & \dots & \mathbf{b}_1^p & \mathbf{b}_3^p+1 & \dots & \mathbf{b}_2^p & \mathbf{b}_4^p+1 & \dots & 1 & \dots & \mathbf{b}_p^p \end{pmatrix}.$$

Note that $b_1 = \mathbf{b}_1^1$, $b_p = \mathbf{b}_p^p$ and $n = \mathbf{b}_1^p$. Also, if $\mathbf{b} = (a, b)$ then $w_{\mathbf{b}} = w_{a,b}$.

For convenience we set $T_{\mathbf{b}} = T_{w_{\mathbf{b}}}$. For example, $T_{a,b} = T_{w_{a,b}}$.

For any $\mathbf{b} = (b_1, \dots, b_p) \in \mathscr{C}_{p,n}$ we define $\mathbf{b}' = (b_p, \dots, b_1)$. Since $w_{a,b}^{-1} = w_{b,a}$ it follows that $w_{\mathbf{b}'} = w_{\mathbf{b}}^{-1}$.

Finally, set $\mathfrak{S}_b = \mathfrak{S}_{b_1} \times \mathfrak{S}_{b_2} \times \dots \times \mathfrak{S}_{b_p}$, which we consider as a subgroup of $\mathfrak{S}_n$ in the obvious way. Similarly, $\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}})$ is a subalgebra of $\mathscr{H}_q(\mathfrak{S}_n)$ via the natural embedding.

The following important property of $v_{\mathbf{b}}$ was established in [25].

Lemma 2.18 ([25, Proposition 2.5])**.** *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $1 \le i, j \le n$, with $i \ne \mathbf{b}_t^p$ for $1 \le t \le p$. Then*

a) $T_i v_{\mathbf{b}} = v_{\mathbf{b}} T_{(i)w_{\mathbf{b}}^{-1}}$, *and*
b) $L_j v_{\mathbf{b}} = v_{\mathbf{b}} L_{(j)w_{\mathbf{b}}^{-1}}$.

Using this fact we can prove the following two results.

Lemma 2.19. *Suppose that $1 \le t \le p$ and let i and j be integers such that $1 \le i, j \le n$ and $i \ne \mathbf{b}^t_\alpha$ for $\alpha = t-p+1, t-p+2, \dots, t$. Then*

$$T_i\Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) = \begin{cases} \Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) T_i, & \text{if } 1 \le i < b_t; \\ \Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) T_{(i)w_{\mathbf{b}\langle t-1\rangle'}}, & \text{if } b_t + 1 \le i < n, \end{cases}$$

$$L_j\Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) = \begin{cases} \Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) L_j, & \text{if } 1 \le j \le b_t; \\ \Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) L_{(j)w_{\mathbf{b}\langle t-1\rangle'}}, & \text{if } b_t + 1 \le j \le n, \end{cases}$$

Proof. For the first equality, if $i \ne b_t$ then using Lemmas 2.11 and 2.9

$$\begin{aligned} T_i Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} &= T_i \mathcal{L}^{(t+1,t+p-1)}_{1,b_t} T_{b_t,n-b_t} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} \\ &= \mathcal{L}^{(t+1,t+p-1)}_{1,b_t} T_i T_{b_t,n-b_t} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} \\ &= \mathcal{L}^{(t+1,t+p-1)}_{1,b_t} T_{b_t,n-b_t} T_{(i)w_{b_t,n-b_t}} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}. \end{aligned}$$

The first claim now follows using Lemma 2.18. For the second claim observe that by Corollary 2.14(b) there exists an $h \in \mathscr{H}_{r,n}$ such that

$$\begin{aligned} L_j Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} &= L_j v^{(t+1,t+p-1)}_{1,b_t} h = v^{(t+1,t+p-1)}_{b_t,n-b_t} L_{(j)w_{b_t,n-b_t}} h \\ &= \mathcal{L}^{(t+1,t+p-1)}_{1,b_t} T_{b_t,n-b_t} L_{(j)w_{b_t,n-b_t}} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} \\ &= Y_t L_{(j)w_{b_t,n-b_t}} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}. \end{aligned}$$

So the result again follows using Lemma 2.18. □

Lemma 2.20. *Suppose that $1 \le t \le p$ and let i and j be integers such that $1 \le i, j \le n$ and $i \ne \mathbf{b}^t_\alpha$ whenever $t-p+1 \le \alpha \le t$. Then*

$$T_i(Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) = \begin{cases} (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{i+\mathbf{b}^{t-1}_1}, & \text{if } 1 \le i < b_t; \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{i-b_t+b_1+\dots+b_{t-2}}, & \text{if } b_t + 1 \le i < \mathbf{b}^t_{t-1}; \\ \quad\vdots & \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{i-\mathbf{b}^t_2}, & \text{if } \mathbf{b}^t_2 + 1 \le i < \mathbf{b}^t_1; \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{(i-\mathbf{b}^t_1)w_{\mathbf{b}'}}, & \text{if } \mathbf{b}^t_1 + 1 \le i < n; \end{cases}$$

$$L_j(Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) = \begin{cases} (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) L_{j+\mathbf{b}^{t-1}_1}, & \text{if } 1 \le j \le b_t; \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) L_{j-b_t+b_1+\dots+b_{t-2}}, & \text{if } b_t + 1 \le j \le \mathbf{b}^t_{t-1}; \\ \quad\vdots & \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) L_{j-\mathbf{b}^t_2}, & \text{if } \mathbf{b}^t_2 + 1 \le j \le \mathbf{b}^t_1; \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) L_{(j-\mathbf{b}^t_1)w_{\mathbf{b}'}}, & \text{if } \mathbf{b}^t_1 + 1 \le j \le n. \end{cases}$$

$$\begin{aligned} T_i(Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) &= (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{(i)w_{(b_t,\dots,b_1)}} \\ L_j(Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) &= (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{(j)w_{(b_t,\dots,b_1)}} \end{aligned}$$

In particular, taking $t = p$, we have

$$\begin{aligned} T_i(Y_p \dots Y_2 Y_1 v_{\mathbf{b}}) &= (Y_p \dots Y_2 Y_1 v_{\mathbf{b}}) T_{(i)w_{\mathbf{b}'}}, \\ L_j(Y_p \dots Y_2 Y_1 v_{\mathbf{b}}) &= (Y_p \dots Y_2 Y_1 v_{\mathbf{b}}) L_{(j)w_{\mathbf{b}'}}. \end{aligned}$$

Proof. This can be proved in exactly the same way as Lemma 2.19. Note that the final claim also follows from Proposition 2.17 using Lemma 2.11. □

The following definition is repeated from (1.1).

Definition 2.21. *Suppose that R is a commutative ring with 1 and set*

$$A(\varepsilon,q,\mathbf{Q})=\prod_{1\le i,j\le d}\prod_{-n<k<n}\prod_{1\le t<p}\big(Q_i-\varepsilon^t q^k Q_j\big).$$

*Then $\mathbf{Q}$ is (ε,q)-**separated** in R if $A(\varepsilon,q,\mathbf{Q})$ is invertible in R.*

Observe that, even though our notation does not reflect this, whether or not $\mathbf{Q}$ is (ε,q)-separated also depends on n and the ring R.

Remark 2.22. When $d=1$, the algebra $\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}})$ can be naturally embedded into $\mathscr{H}_{r,n}$ as a subalgebra; see [22]. In that case, the condition of being (ε,q)-separated means that $\prod_{|k|<n,1\le t<p}\big(1-\varepsilon^t q^k\big)$ is invertible.

Fix $\mathbf{b}\in\mathscr{C}_{p,n}$ and set $V_{\mathbf{b}}=v_{\mathbf{b}}\mathscr{H}_{r,n}$ and $\mathscr{H}_{d,\mathbf{b}}=\mathscr{H}_{d,b_1}(\varepsilon\mathbf{Q})\otimes\cdots\otimes\mathscr{H}_{d,b_p}(\varepsilon^p\mathbf{Q})$. Then an important result from [25] is the following.

Proposition 2.23 ([25, Proposition 2.15])). *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$ and that $\mathbf{Q}$ is (ε,q)-separated if $d>1$. Then:*

a) *$\mathscr{H}_{d,\mathbf{b}}$ acts faithfully on $V_{\mathbf{b}}$ from the left and $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}})\cong\mathscr{H}_{d,\mathbf{b}}$.*
b) *$V_{\mathbf{b}}$ is projective as an $\mathscr{H}_{r,n}$-module and $\bigoplus_{\mathbf{b}\in\mathscr{C}_{p,n}}V_{\mathbf{b}}$ is a progenerator for $\mathscr{H}_{r,n}$.*

To describe the action of $\mathscr{H}_{d,\mathbf{b}}$ on $V_{\mathbf{b}}$ given a permutation $w=s_{i_1}\dots s_{i_k}\in\mathfrak{S}_n$ and an integer $c\in\mathbb{N}$ such that $i_j+c<n$, for $1\le j\le k$, define $w^{\langle c\rangle}=s_{i_1+c}\dots s_{i_k+c}$. Then $w^{\langle c\rangle}\in\mathfrak{S}_n$. Note that this is compatible with our previous definition of $w_{a,b}^{\langle c\rangle}$.

Define $\Theta_{\mathbf{b}}$ to be the 'natural inclusion map' $\mathscr{H}_{d,\mathbf{b}}\hookrightarrow\mathscr{H}_{r,n}$. That is, $\Theta_{\mathbf{b}}$ is the R-linear map determined by

$$\begin{aligned}\Theta_{\mathbf{b}}&\Big(\big(L_1^{a_{1,1}}\dots L_{b_1}^{a_{1,b_1}}T_{x_1}\big)\otimes\big(L_1^{a_{2,1}}\dots L_{b_2}^{a_{2,b_2}}T_{x_2}\big)\otimes\cdots\otimes\big(L_1^{a_{p,1}}\dots L_{b_p}^{a_{p,b_p}}T_{x_p}\big)\Big)\\
&=\big(L_1^{a_{1,1}}\dots L_{b_1}^{a_{1,b_1}}T_{x_1'}\big)\big(L_{\mathbf{b}_1^1+1}^{a_{2,1}}\dots L_{\mathbf{b}_1^1+b_2}^{a_{2,b_2}}T_{x_2'}\big)\cdots\big(L_{\mathbf{b}_1^{p-1}+1}^{a_{p,1}}\dots L_{\mathbf{b}_1^{p-1}+b_p}^{a_{p,b_p}}T_{x_p'}\big)\\
&=\big(L_1^{a_{1,1}}\dots L_{b_1}^{a_{1,b_1}}\big)\big(L_{\mathbf{b}_1^1+1}^{a_{2,1}}\dots L_{\mathbf{b}_1^1+b_2}^{a_{2,b_2}}\big)\cdots\big(L_{\mathbf{b}_1^{p-1}+1}^{a_{p,1}}\dots L_{n}^{a_{p,b_p}}\big)T_{x_1'}T_{x_2'}\dots T_{x_p'},\end{aligned}$$

for all $x_t\in\mathfrak{S}_{b_t}$ and $0\le a_{j,t}<d$, for $1\le t\le p$ and $1\le j\le b_t$, and where $x_t':=x_t^{\langle\mathbf{b}_1^{t-1}\rangle}$, for $1\le t\le p$. The second equality follows because all of these terms commute. Thus, we have $x_1'=x_1$ and $\Theta_{\mathbf{b}}(T_{x_1}\otimes\cdots\otimes T_{x_p})=T_w$, where $w=x_1x_2^{\langle\mathbf{b}_1^1\rangle}\dots x_p^{\langle\mathbf{b}_1^{p-1}\rangle}\in\mathfrak{S}_{\mathbf{b}}$, for $x_t\in\mathfrak{S}_{b_t}$. We emphasize that $\Theta_{\mathbf{b}}$ is an R-module homomorphism but *not* a ring homomorphism.

Similarly, define $\widehat{\Theta}_{\mathbf{b}}$ to be the R-linear map $\widehat{\Theta}_{\mathbf{b}}:\mathscr{H}_{d,\mathbf{b}}\longrightarrow\mathscr{H}_{r,n}$ determined by

$$\begin{aligned}\widehat{\Theta}_{\mathbf{b}}&\Big(\big(L_1^{a_{1,1}}\dots L_{b_1}^{a_{1,b_1}}T_{x_1}\big)\otimes\big(L_1^{a_{2,1}}\dots L_{b_2}^{a_{2,b_2}}T_{x_2}\big)\otimes\cdots\otimes\big(L_1^{a_{p,1}}\dots L_{b_p}^{a_{p,b_p}}T_{x_p}\big)\Big)\\
&=\big(L_1^{a_{p,1}}\dots L_{b_p}^{a_{p,b_p}}T_{x_p''}\big)\cdots\big(L_{\mathbf{b}_3^p+1}^{a_{2,1}}\dots L_{\mathbf{b}_2^p}^{a_{2,b_2}}T_{x_2''}\big)\big(L_{\mathbf{b}_2^p+1}^{a_{1,1}}\dots L_{\mathbf{b}_1^p}^{a_{1,b_1}}T_{x_1''}\big)\\
&=\big(L_1^{a_{p,1}}\dots L_{b_p}^{a_{p,b_p}}\big)\dots\big(L_{\mathbf{b}_3^p+1}^{a_{2,1}}\dots L_{\mathbf{b}_2^p}^{a_{2,b_2}}\big)\big(L_{\mathbf{b}_2^p+1}^{a_{1,1}}\dots L_{\mathbf{b}_1^p}^{a_{1,b_1}}\big)T_{x_1''}T_{x_2''}\dots T_{x_p''},\end{aligned}$$

where the x_t and $a_{t,j}$ are as before and $x_t'':=w_{\mathbf{b}}^{-1}x_t^{\langle\mathbf{b}_1^{t-1}\rangle}w_{\mathbf{b}}=w_{\mathbf{b}}^{-1}x_t'w_{\mathbf{b}}$. In particular, $x_p''=x_p$ and $x_1''x_2''\dots x_p''=w_{\mathbf{b}}^{-1}\big(x_1x_2^{\langle\mathbf{b}_1^1\rangle}\dots x_p^{\langle\mathbf{b}_1^{p-1}\rangle}\big)w_{\mathbf{b}}\in\mathfrak{S}_{\mathbf{b}'}$.

Given these definitions, the proof of Proposition 2.23(a), that is, of [25, Proposition 2.15], shows that $h \in \mathscr{H}_{d,\mathbf{b}}$ acts on $V_{\mathbf{b}}$ as left multiplication by $\widehat{\Theta}_{\mathbf{b}}(h)$. Moreover,

$$\widehat{\Theta}_{\mathbf{b}}(h)v_{\mathbf{b}} = v_{\mathbf{b}}\Theta_{\mathbf{b}}(h), \qquad \text{for all } h \in \mathscr{H}_{d,\mathbf{b}}, \tag{2.24}$$

by Lemma 2.18. Typically, if $h \in \mathscr{H}_{d,\mathbf{b}}$ then we write $h \cdot v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(h)v_{\mathbf{b}}$ in what follows. Thus,

$$h \cdot v_{\mathbf{b}} = v_{\mathbf{b}}\Theta_{\mathbf{b}}(h), \qquad \text{for all } h \in \mathscr{H}_{d,\mathbf{b}},$$

for $h \in \mathscr{H}_{d,\mathbf{b}}$.

The following lemma introduces the elements $z_{\mathbf{b}}$. These elements play a central role in the proofs of all of our Main Theorems from the introduction.

Lemma 2.25. *Suppose that $\mathbf{Q}$ is (ε, q)-separated and let $\mathbf{b} \in \mathscr{C}_{p,n}$. Then there exists a unique element $z_{\mathbf{b}}$ in $\mathscr{H}_{d,\mathbf{b}}$ such that*

$$z_{\mathbf{b}} \cdot v_{\mathbf{b}} = Y_pY_{p-1}\dots Y_2Y_1v_{\mathbf{b}} = v_{\mathbf{b}}\Theta_{\mathbf{b}}(z_{\mathbf{b}}).$$

Moreover, $z_{\mathbf{b}}$ belongs to the centre of $\mathscr{H}_{d,\mathbf{b}}$.

Proof. By Proposition 2.17, left multiplication by $Y_p \dots Y_2Y_1$ defines a homomorphism in $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}})$. Therefore, there exists a unique element $z_{\mathbf{b}}$ in $\mathscr{H}_{d,\mathbf{b}}$ such that

$$Y_pY_{p-1}\dots Y_2Y_1v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}})v_{\mathbf{b}} = v_{\mathbf{b}}\Theta_{\mathbf{b}}(z_{\mathbf{b}})$$

by Proposition 2.23(a) and (2.24).

It remains to show that $z_{\mathbf{b}}$ is central in $\mathscr{H}_{d,\mathbf{b}}$. As $\mathscr{H}_{d,\mathbf{b}}$ acts faithfully on $V_{\mathbf{b}}$, it is enough to show that $\widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}}h)v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(hz_{\mathbf{b}})v_{\mathbf{b}}$, for all $h \in \mathscr{H}_{d,\mathbf{b}}$. By Lemma 2.18,

$$\widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}}h)v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}})\widehat{\Theta}_{\mathbf{b}}(h)v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}})v_{\mathbf{b}}\Theta_{\mathbf{b}}(h) = Y_p \dots Y_2Y_1v_{\mathbf{b}}\Theta_{\mathbf{b}}(h).$$

Applying (the last statements in) Lemma 2.20, shows that

$$Y_p \dots Y_2Y_1v_{\mathbf{b}}\Theta_{\mathbf{b}}(h) = \widehat{\Theta}_{\mathbf{b}}(h)Y_p \dots Y_2Y_1v_{\mathbf{b}}, = \widehat{\Theta}_{\mathbf{b}}(h)\widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}})v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(hz_{\mathbf{b}})v_{\mathbf{b}},$$

as required. □

2.5. **A Morita equivalence for $\mathscr{H}_{r,n}$.** In this section we give a new description of the Morita equivalence of Theorem 2.5 which will be useful for proving the first half of Theorem B. In particular, in this section we will show that $z_{\mathbf{b}}$ is an invertible element of $\mathscr{H}_{d,\mathbf{b}}$.

By Proposition 2.23(b), $V_{\mathbf{b}}$ is a projective $\mathscr{H}_{r,n}$-module. Let $\mathscr{H}_{r,n}(\mathbf{b})$ be the smallest two-sided ideal of $\mathscr{H}_{r,n}$ which contains $V_{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ as a direct summand. By [12, Theorem 1.1] the Morita equivalence of Theorem 2.5 is induced by equivalences

$$\mathtt{H}_{\mathbf{b}} : \text{Mod-}\mathscr{H}_{d,\mathbf{b}} \xrightarrow[\text{Morita}]{\simeq} \text{Mod-}\mathscr{H}_{r,n}(\mathbf{b})$$

given by $\mathtt{H}_{\mathbf{b}}(X) = X \otimes_{\mathscr{H}_{d,\mathbf{b}}} V_{\mathbf{b}}$. Hence, by Proposition 2.23(a) and the general theory of Morita equivalences (*cf.* [5, §2.2]), we have the following.

Lemma 2.26 ((*cf.* [12, Corollary 4.9]))**.** *Suppose that $\mathbf{Q}$ is (ε, q)-separated in R and let X be a right ideal of $\mathscr{H}_{d,\mathbf{b}}$. Then, as right $\mathscr{H}_{r,n}$-modules,*

$$\mathtt{H}_{\mathbf{b}}(X) \cong \widehat{\Theta}_{\mathbf{b}}(X)V_{\mathbf{b}}.$$

We next show that $\mathtt{H}_{\mathbf{b}}$ can be realised as induction from a subalgebra of $\mathscr{H}_{r,n}$. To do this we need to produce a subalgebra of $\mathscr{H}_{r,n}$ which is isomorphic to $\mathscr{H}_{d,\mathbf{b}}$.

Before we state this result, given a sequence $\mathbf{b} = (b_1, \dots, b_p) \in \mathscr{C}_{p,n}$ define

$$u_{\mathbf{b}}^+(\mathbf{Q}) = \mathcal{L}^{(2)}_{1,\mathbf{b}_1^1}\mathcal{L}^{(3)}_{1,\mathbf{b}_1^2}\dots\mathcal{L}^{(p)}_{1,\mathbf{b}_1^{p-1}} \quad\text{and}\quad u_{\mathbf{b}}^-(\mathbf{Q}) = \mathcal{L}^{(p-1)}_{1,\mathbf{b}_p^p}\dots\mathcal{L}^{(2)}_{1,\mathbf{b}_3^p}\mathcal{L}^{(1)}_{1,\mathbf{b}_2^p}. \tag{2.27}$$

In the notation of [11, Definition 3.1], $u_{\mathbf{b}}^+(\mathbf{Q}) = u^+_{\boldsymbol{\omega}_{\mathbf{b}}}$, where $\boldsymbol{\omega}_{\mathbf{b}} = ({\boldsymbol{\omega}_{\mathbf{b}}}^{(1)}, \cdots, {\boldsymbol{\omega}_{\mathbf{b}}}^{(r)})$ is the multipartition

$${\boldsymbol{\omega}_{\mathbf{b}}}^{(s)} = \begin{cases} (1^{b_\alpha}), & \text{if } s = d\alpha \text{ for some } \alpha,\\ (0), & \text{otherwise.}\end{cases}$$

Hereafter, we write $u_{\mathbf{b}}^\pm = u_{\mathbf{b}}^\pm(\mathbf{Q})$.

Taking $j = 1$ and $j = p$ in Proposition 2.13, respectively, we can write $v_{\mathbf{b}} = v_{\mathbf{b}}^+u_{\mathbf{b}}^+ = u_{\mathbf{b}}^-v_{\mathbf{b}}^-$ where

$$v_{\mathbf{b}}^+ = \mathcal{L}^{(1,p-1)}_{1,b_p}T_{b_p,\mathbf{b}_1^{p-1}}\mathcal{L}^{(1,p-2)}_{1,b_{p-1}}T_{b_{p-1},\mathbf{b}_1^{p-2}}\dots\mathcal{L}^{(1,1)}_{1,b_2}T_{b_2,\mathbf{b}_1^1}$$

and

$$v_{\mathbf{b}}^- = T_{\mathbf{b}_p^p,b_{p-1}}\mathcal{L}^{(p,p)}_{1,b_{p-1}}\dots T_{\mathbf{b}_3^p,b_2}\mathcal{L}^{(3,p)}_{1,b_2}T_{\mathbf{b}_2^p,b_1}\mathcal{L}^{(2,p)}_{1,b_1}.$$

Lemma 2.28. *Suppose that $\mathbf{Q}$ is (ε, q)-separated in R. Let $\mathbf{b} \in \mathscr{C}_{p,n}$. Then $z_{\mathbf{b}}$ is invertible in $\mathscr{H}_{d,\mathbf{b}}$.*

Proof. The module $V_{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ is a projective submodule of $\mathscr{H}_{r,n}$-module by Proposition 2.23(b), so $V_{\mathbf{b}} = e\mathscr{H}_{r,n}$ for some idempotent $e \in \mathscr{H}_{r,n}$. Therefore, $V_{\mathbf{b}} = e\mathscr{H}_{r,n} = e^2\mathscr{H}_{r,n} \subseteq e\mathscr{H}_{r,n}e\mathscr{H}_{r,n} = V_{\mathbf{b}}^2 \subseteq e\mathscr{H}_{r,n} = V_{\mathbf{b}}$ so that $V_{\mathbf{b}} = V_{\mathbf{b}}^2$. Therefore, using the formulae for $v_{\mathbf{b}}$ given before the Lemma,

$$\begin{aligned} V_{\mathbf{b}} = (V_{\mathbf{b}})^2 &= v_{\mathbf{b}}\mathscr{H}_{r,n}v_{\mathbf{b}}\mathscr{H}_{r,n} = v_{\mathbf{b}}^+\big(u_{\mathbf{b}}^+\mathscr{H}_{r,n}u_{\mathbf{b}}^-\big)v_{\mathbf{b}}^-\mathscr{H}_{r,n}\\ &= v_{\mathbf{b}}^+\big(u_{\mathbf{b}}^+T_{\mathbf{b}}u_{\mathbf{b}}^-\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}})\big)v_{\mathbf{b}}^-\mathscr{H}_{r,n},\end{aligned}$$

where the last equality follows by Du and Rui [13, Proposition 3.1(a)]. Lemma 2.9 shows that $\mathscr{H}_q(\mathfrak{S}_b)v_{\mathbf{b}}^- = v_{\mathbf{b}}^-\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}'})$. Hence,

$$V_{\mathbf{b}} = v_{\mathbf{b}}T_{\mathbf{b}}u_{\mathbf{b}}^-v_{\mathbf{b}}^-\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}'})\mathscr{H}_{r,n} \subseteq v_{\mathbf{b}}T_{\mathbf{b}}v_{\mathbf{b}}\mathscr{H}_{r,n} = z_{\mathbf{b}}\cdot V_{\mathbf{b}} \subseteq V_{\mathbf{b}},$$

by Proposition 2.17 and Lemma 2.25. Therefore, $V_{\mathbf{b}} = z_{\mathbf{b}} \cdot V_{\mathbf{b}}$ so the endomorphism of $V_{\mathbf{b}}$ given by left multiplication by $z_{\mathbf{b}}$ has a right inverse in $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}})$. Consequently, $z_{\mathbf{b}}$ has a right inverse in $\mathscr{H}_{d,\mathbf{b}}$ by Proposition 2.23(a). Hence, $z_{\mathbf{b}}$ is invertible in $\mathscr{H}_{d,\mathbf{b}}$ since it is central. $\square$

Corollary 2.29. *Let $\mathbf{b} \in \mathscr{C}_{p,n}$, $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ and $\mathtt{t} \in \mathbb{Z}$. Suppose that $\mathbf{Q}$ is (ε, q)-separated over the field K. Then $V^{(t)}_{\mathbf{b}\langle t\rangle} \cong V^{(t+1)}_{\mathbf{b}\langle t+1\rangle}$.*

Proof. It is enough to consider the case where $t = 0$. By Lemma 2.25 left multiplication by Y_1 induces an $\mathscr{H}_{r,n}$-module homomorphism from $V_{\mathbf{b}}$ to $V^{(1)}_{\mathbf{b}\langle 1\rangle}$. This map is an isomorphism because left multiplication by $Y_p \dots Y_1$ is invertible by Lemma 2.28 (and Lemma 2.25). $\square$

Under the conditions of Lemma 2.28 we can make the following definition.

Definition 2.30. *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and that* $\mathbf{Q}$ *is* (ε,q)*-separated in* R. *Let* $e_{\mathbf{b}}=z_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}}\in V_{\mathbf{b}}$ *and define*

$$\widehat{\mathscr{H}_{d,\mathbf{b}}}=\{\,h\cdot e_{\mathbf{b}}\mid h\in\mathscr{H}_{d,\mathbf{b}}\,\}=\{\,e_{\mathbf{b}}\Theta_{\mathbf{b}}(h)\mid h\in\mathscr{H}_{d,b}\,\}\subseteq V_b.$$

Quite surprisingly, $\widehat{\mathscr{H}_{d,\mathbf{b}}}$ is something like a parabolic subalgebra of $\mathscr{H}_{r,n}$.

Theorem 2.31. *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and that* $\mathbf{Q}$ *is* (ε,q)*-separated. Then:*

a) $e_{\mathbf{b}}$ *is an idempotent in* $\mathscr{H}_{r,n}$ *and* $V_{\mathbf{b}}=e_{\mathbf{b}}\mathscr{H}_{r,n}$.
b) $\widehat{\mathscr{H}_{d,\mathbf{b}}}$ *is a unital subalgebra of* $\mathscr{H}_{r,n}$ *with identity element* $e_{\mathbf{b}}$.
c) *The map* $\mathscr{H}_{d,\mathbf{b}}\longrightarrow\widehat{\mathscr{H}_{d,\mathbf{b}}}; h\mapsto h\cdot e_{\mathbf{b}}$ *is an algebra isomorphism.*

Proof. Suppose that $x,y\in\mathscr{H}_{d,\mathbf{b}}$. Then using the definitions, (2.24) and Lemma 2.25 we have that

$$\begin{aligned}(x\cdot e_{\mathbf{b}})(y\cdot e_{\mathbf{b}})&=(xz_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}})(yz_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}})=xz_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}}v_{\mathbf{b}}\Theta_{\mathbf{b}}(yz_{\mathbf{b}}^{-1})T_{\mathbf{b}}\\&=xz_{\mathbf{b}}^{-1}z_{\mathbf{b}}\cdot v_{\mathbf{b}}\Theta_{\mathbf{b}}(yz_{\mathbf{b}}^{-1})T_{\mathbf{b}}=x\cdot v_{\mathbf{b}}\Theta_{\mathbf{b}}(yz_{\mathbf{b}}^{-1})T_{\mathbf{b}}\\&=xyz_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}}=(xy)\cdot e_{\mathbf{b}}.\end{aligned}$$

Taking $x=y=1_{\mathscr{H}_{d,\mathbf{b}}}$ shows that $e_{\mathbf{b}}$ is an idempotent in $\mathscr{H}_{r,n}$. As $\mathscr{H}_{d,\mathbf{b}}$ acts faithfully on $V_{\mathbf{b}}$ by Proposition 2.23(a), all of the claims now follow. □

Theorem 2.31 says that the natural inclusion map $\Theta_{\mathbf{b}}:\mathscr{H}_{d,\mathbf{b}}\hookrightarrow\mathscr{H}_{r,n}$ is an inclusion of algebras when it is composed with left multiplication by $e_{\mathbf{b}}$. Note that, in general, the image of $\Theta_{\mathbf{b}}$ is *not* a subalgebra of $\mathscr{H}_{r,n}$.

Combining Theorem 2.31 and Lemma 2.26 gives a second description of the Morita equivalence $\mathtt{H}_{\mathbf{b}}$. If A is a subalgebra of an algebra B then let $\uparrow_A^B$ be the corresponding induction functor.

Corollary 2.32. *Suppose that* $\mathbf{Q}$ *is* (ε,q)*-separated and that* X *is a right* $\mathscr{H}_{d,\mathbf{b}}$ *module, where* $\mathbf{b}\in\mathscr{C}_{p,n}$. *Then*

$$\mathtt{H}_{\mathbf{b}}(X)\cong(X\cdot e_{\mathbf{b}})\uparrow_{\widehat{\mathscr{H}_{d,\mathbf{b}}}}^{\mathscr{H}_{r,n}}=X\cdot e_{\mathbf{b}}\otimes_{\widehat{\mathscr{H}_{d,\mathbf{b}}}}\mathscr{H}_{r,n}.$$

2.6. **Comparing trace forms on $V_{\mathbf{b}}$.** Theorem 2.31 shows how to realize $\mathscr{H}_{d,\mathbf{b}}$ as a subalgebra of $\mathscr{H}_{r,n}$. The aim of this section is to use this result to prove a comparison theorem for the natural trace forms on $\mathscr{H}_{r,n}$ and $\mathscr{H}_{d,\mathbf{b}}$. This is one of the key steps in proving Theorem B from the introduction because the Schur element $s_{\boldsymbol{\lambda}}$ from (1.4) can be computed from the trace of the primitive idempotents for the Specht module $S(\boldsymbol{\lambda})$ in the semisimple case.

Recall that a trace form on an R-algebra A is a linear map $\operatorname{tr}:A\longrightarrow R$ such that $\operatorname{tr}(ab)=\operatorname{tr}(ba)$, for all $a,b\in A$. The form tr is non-degenerate if whenever $0\neq a\in A$ then $\operatorname{tr}(ab)\neq 0$ for some $b\in A$.

By [27] the Hecke algebras $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$ are both equipped with 'canonical' non-degenerate trace forms $\operatorname{Tr}_{\mathbf{b}}$ and Tr, respectively. The aim of this subsection is to compare these two trace forms. More precisely, we show that

$$\operatorname{Tr}(h\cdot v_{\mathbf{b}}T_{\mathbf{b}})=\operatorname{Tr}_{\mathbf{b}}(h)\operatorname{Tr}(v_{\mathbf{b}}T_{\mathbf{b}}),$$

for all $h\in\mathscr{H}_{d,\mathbf{b}}$. This result will be used in the next section to compute the scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$ from the introduction.

The trace form $\operatorname{Tr}:\mathscr{H}_{r,n}\longrightarrow R$ on $\mathscr{H}_{r,n}$ is the R-linear map determined by

$$\operatorname{Tr}(L_1^{a_1}\dots L_n^{a_n}T_xT_y)=\begin{cases}q^{\ell(x)}, & \text{if } a_1=\dots=a_n=0 \text{ and } x=y^{-1},\\ 0, & \text{otherwise.}\end{cases}\tag{2.33}$$

(This equation completely determines Tr by Lemma 2.7.) The trace form $\operatorname{Tr}_{\mathbf{b}}$ on $\mathscr{H}_{d,\mathbf{b}}$ is defined similarly. Comparing these two trace forms requires more technical calculations with the elements $v_{\mathbf{b}}$.

Before Lemma 2.28 we noted that $v_{\mathbf{b}}=v_{\mathbf{b}}^+u_{\mathbf{b}}^+$, for some element $v_{\mathbf{b}}^+$. To compare the trace forms Tr and $\operatorname{Tr}_{\mathbf{b}}$ we need a different expression for $v_{\mathbf{b}}^+$. To state this, let $\mathcal{H}_m^L$ be the R-submodule of $\mathscr{H}_{r,n}$ spanned by the elements

$$\{\,T_wL_1^{a_1}\dots L_{m-1}^{a_{m-1}}\mid 0\le a_1,\dots,a_{m-1}<r \text{ and } w\in\mathfrak{S}_m\,\}.$$

Note that $\mathcal{H}_m^L$ is not, in general, a subalgebra of $\mathscr{H}_{r,n}$.

The proof of the next result is not particularly pretty so we defer it until Lemma A6. Recall from Section 2.4 that $\mathbf{b}'=(b_p,\dots,b_1)$ if $\mathbf{b}=(b_1,\dots,b_p)$.

Lemma 2.34. *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$. Then*

$$v_{\mathbf{b}}^+=T_{\mathbf{b}'}\Big(\mathcal{L}^{(1)}_{\mathbf{b}_1^1+1,n}\mathcal{L}^{(2)}_{\mathbf{b}_1^2+1,n}\dots\mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n}+\sum_{l=1}^{p-1}\sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}}\sum_{e=1}^{dl}h_{l,m,e}L_m^e\Big)$$

for some $h_{l,m,e}\in\mathcal{H}_m^L$.

Using this result we can prove the promised comparison theorem for Tr and $\operatorname{Tr}_{\mathbf{b}}$.

Theorem 2.35. *Suppose that $b\in\mathscr{C}_{p,n}$. Then*

$$\operatorname{Tr}(h\cdot v_{\mathbf{b}}T_{\mathbf{b}})=\operatorname{Tr}_{\mathbf{b}}(h)\operatorname{Tr}(v_{\mathbf{b}}T_{\mathbf{b}}),$$

for all $h\in\mathscr{H}_{d,\mathbf{b}}$.

Proof. By linearity, it is enough to let h run over a basis of $\mathscr{H}_{d,\mathbf{b}}$. Let

$$\mathfrak{B}_{\mathbf{b}}=\{\,L_1^{a_{1,1}}\dots L_{b_1}^{a_{1,b_1}}T_{x_1}\otimes\dots\otimes L_1^{a_{p,1}}\dots L_{b_p}^{a_{p,b_p}}T_{x_p}\mid 0\le a_{i,t}<d \text{ and } x_t\in\mathfrak{S}_{b_t}\,\}$$

be the basis of $\mathscr{H}_{d,\mathbf{b}}$ from Lemma 2.7. Then it is enough to show that

$$\operatorname{Tr}(h\cdot v_{\mathbf{b}}T_{\mathbf{b}})=\operatorname{Tr}_{\mathbf{b}}(h)\operatorname{Tr}(v_{\mathbf{b}}T_{\mathbf{b}}),\qquad \text{for all } h\in\mathfrak{B}_{\mathbf{b}}.$$

If $h=1_{\mathscr{H}_{d,\mathbf{b}}}$ there is nothing to prove. Therefore, by (2.33) it remains to show that $\operatorname{Tr}(h\cdot v_{\mathbf{b}}T_{\mathbf{b}})=0$ whenever $1_{\mathscr{H}_{d,\mathbf{b}}}\ne h\in\mathfrak{B}_{\mathbf{b}}$. For the rest of the proof fix such an h. Write $h=L_1^{a_{1,1}}\dots L_{b_1}^{a_{1,b_1}}T_{x_1}\otimes\dots\otimes L_1^{a_{p,1}}\dots L_{b_p}^{a_{p,b_p}}T_{x_p}$, where $0\le a_{j,t}<d$ and $x_t\in\mathfrak{S}_{b_t}$, and set

$$h'=\Theta_{\mathbf{b}}(h)=L_1^{a_{1,1}}\dots L_{\mathbf{b}_1^1}^{a_{1,b_1}}L_{\mathbf{b}_1^1+1}^{a_{2,1}}\dots L_{\mathbf{b}_1^2}^{a_{2,b_2}}\dots L_{\mathbf{b}_1^{p-1}+1}^{a_{p,1}}\dots L_{\mathbf{b}_1^p}^{a_{p,b_p}}T_x,$$

where $x=x_1x_2^{\langle\mathbf{b}_1^1\rangle}\dots x_p^{\langle\mathbf{b}_1^{p-1}\rangle}$.

Recall from before Lemma 2.28 that $v_{\mathbf{b}} = v_{\mathbf{b}}^{+}u_{\mathbf{b}}^{+}$. Therefore, using Lemma 2.34 and the fact that Tr is a trace form,

$$\begin{aligned}\operatorname{Tr}(h\cdot v_{\mathbf{b}}T_{\mathbf{b}}) &= \operatorname{Tr}(v_{\mathbf{b}}h'T_{\mathbf{b}}) = \operatorname{Tr}(v_{\mathbf{b}}^{+}u_{\mathbf{b}}^{+}h'T_{\mathbf{b}})\\ &= \operatorname{Tr}(T_{\mathbf{b}'}\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+}h'T_{\mathbf{b}}) + \sum_{l=1}^{p-1}\sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}}\sum_{e=1}^{dl}\operatorname{Tr}(T_{\mathbf{b}'}h_{l,m,e}L_m^e u_{\mathbf{b}}^{+}h'T_{\mathbf{b}})\\ &= \operatorname{Tr}(\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+}h'T_{\mathbf{b}}T_{\mathbf{b}'}) + \sum_{l=1}^{p-1}\sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}}\sum_{e=1}^{dl}\operatorname{Tr}(L_m^e u_{\mathbf{b}}^{+}h'T_{\mathbf{b}}T_{\mathbf{b}'}h_{l,m,e}),\end{aligned}$$

where $h_{l,m,e}\in\mathcal{H}_m^L$ and $\hat{u}_{\mathbf{b}}^{-} := \mathcal{L}^{(1)}_{\mathbf{b}_1^1+1,n}\mathcal{L}^{(2)}_{\mathbf{b}_1^2+1,n}\cdots\mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n}$. Fix a triple (l,m,e), from the sum, with $1\le l<p$, $\mathbf{b}_1^l < m \le \mathbf{b}_1^{l+1}$ and $1\le e\le dl$. By assumption, L_m appears in h' with exponent $0\le a_{l+1,m'}<d$, where $m=\mathbf{b}_1^l+m'$. Therefore, $L_m^e u_{\mathbf{b}}^{+}h'T_{\mathbf{b}}T_{\mathbf{b}'}h_{l,m,e}$ is a linear combination of terms of the form $L_m^e u_{\mathbf{b}}^{+}f_1(L)T_w f_2(L)$, where $w\in\mathfrak{S}_n$, $f_1(L)$ is a a polynomial in $L_1,\dots,L_n$ of degree at most $a_{l+1,m'}<d$ as a polynomial in L_m, and where $f_2(L)$ is a polynomial in $L_1,\dots,L_{m-1}$. As Tr is a trace form,

$$\operatorname{Tr}(L_m^e u_{\mathbf{b}}^{+}f_1(L)T_w f_2(L)) = \operatorname{Tr}(f_2(L)L_m^e u_{\mathbf{b}}^{+}f_1(L)T_w).$$

Now, considered as a polynomial in L_m, $f_2(L)L_m^e u_{\mathbf{b}}^{+}f_1(L)$ is a polynomial with zero constant term (since $e>0$) and degree

$$0 < f := e + d(p-l-1) + a_{l+1,m'} < d(p-1)+d = r.$$

By the same argument, if $m<k\le n$ then L_k appears in $f_2(L)L_m^e u_{\mathbf{b}}^{+}f_1(L)$ with exponent at most $d(p-l'_k-1)+a_{l_k+1,k'} < d(p-1) < r$, where $k=\mathbf{b}_1^{l_k-1}+k'$ and $1\le k'\le b_{l_k}$. If $k<m$ then L_k could appear in $f_2(L)L_m^e u_{\mathbf{b}}^{+}f_1(L)$ with exponent greater than $r-1$, however, by Lemma 2.8 this will not affect the exponents of $L_m,\dots,L_n$ when rewrite this term as a linear combination of Ariki-Koike basis elements. Hence, L_m^f is a left divisor of $f_2(L)L_m^e u_{\mathbf{b}}^{+}f_1(L)$ when it is written as a linear combination of Ariki-Koike basis elements. Consequently, $\operatorname{Tr}(f_2(L)L_m^e u_{\mathbf{b}}^{+}f_1(L)) = 0$ by (2.33). Therefore, $\operatorname{Tr}(L_m^e u_{\mathbf{b}}^{+}h'T_{\mathbf{b}}T_{\mathbf{b}'}h_{l,m,e}) = 0$ so that $\operatorname{Tr}(h\cdot v_{\mathbf{b}}h'T_{\mathbf{b}}) = \operatorname{Tr}(v_{\mathbf{b}}h'T_{\mathbf{b}}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+}h'T_{\mathbf{b}}T_{\mathbf{b}'})$.

Now consider $\operatorname{Tr}(\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+}h'T_{\mathbf{b}}T_{\mathbf{b}'})$. By definition,

$$\begin{aligned}\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+}h' &= \mathcal{L}^{(1)}_{\mathbf{b}_1^1+1,n}\mathcal{L}^{(2)}_{\mathbf{b}_1^2+1,n}\cdots\mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n}\cdot\mathcal{L}^{(2)}_{1,\mathbf{b}_1^1}\mathcal{L}^{(3)}_{1,\mathbf{b}_1^2}\cdots\mathcal{L}^{(p)}_{1,\mathbf{b}_1^{p-1}}h'\\ &= \prod_{i=1}^{p}\mathcal{L}^{(i)}_{1,\mathbf{b}_1^{i-1}}\mathcal{L}^{(i)}_{\mathbf{b}_1^i+1,n}\cdot h'.\end{aligned}$$

If $a_{l,m'}\ne 0$, for some l and m', then $L_m^{a_{l,m'}}$ divides h', where $m=\mathbf{b}_1^{l-1}+m'$ as above. By the argument above $\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+}h'$, when considered as a polynomial in L_m, is a polynomial with zero constant term and degree strictly less than r. Therefore,

$$\operatorname{Tr}(h\cdot v_{\mathbf{b}}T_{\mathbf{b}}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+}h'T_{\mathbf{b}}T_{\mathbf{b}'}) = 0,$$

as required. It remains, then, to consider the cases when $a_{l,m'}=0$, for $1\le l\le p$ and $1\le m'\le b_l$. That is, when $h'=T_x$ for some $1\ne x\in\mathfrak{S}_{\mathbf{b}}$. By (2.33), in this case we have

$$\operatorname{Tr}(h\cdot v_{\mathbf{b}}T_{\mathbf{b}}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+}T_xT_{\mathbf{b}}T_{\mathbf{b}'}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^{-}u_{\mathbf{b}}^{+})\operatorname{Tr}(T_xT_{\mathbf{b}}T_{\mathbf{b}'}) \tag{2.36}$$

Recall that $w_{\mathbf{b}}$ is a distinguished coset representative for $\mathfrak{S}_{\mathbf{b}}$, so that $\ell(xw_{\mathbf{b}}) = \ell(x) + \ell(w_{\mathbf{b}})$. Therefore, $\operatorname{Tr}(T_x T_{\mathbf{b}} T_{\mathbf{b}'}) = \operatorname{Tr}(T_{xw_{\mathbf{b}}} T_{\mathbf{b}'}) = 0$ by (2.33) since $x \neq 1$. Hence, $\operatorname{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = 0$, completing the proof. □

We can improve on Theorem 2.35 by explicitly computing $\operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}})$. In fact, in proving the theorem we have essentially already done this. To state the result, given $\mathbf{b} \in \mathscr{C}_{p,n}$ set $\alpha(\mathbf{b}) = \sum_{i=1}^{p} ib_i \in \mathbb{N}$.

Corollary 2.37. *Suppose that $b \in \mathscr{C}_{p,n}$. Then*

$$\operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}) = (-1)^{dn(p-1)} q^{\ell(w_{\mathbf{b}})} \varepsilon^{\frac{1}{2}rn(p-1)-d\alpha(\mathbf{b})} (Q_1 \dots Q_d)^{n(p-1)}.$$

Proof. By (2.36), and (2.33), we have that

$$\operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}) = \operatorname{Tr}(1_{\mathscr{H}_{d,\mathbf{b}}} \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+) \operatorname{Tr}(T_{\mathbf{b}'} T_{\mathbf{b}}) = q^{\ell(w_{\mathbf{b}})} \operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+).$$

Now, $\operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+)$ is just the constant term of $\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+$ by (2.33). Therefore,

$$\begin{aligned}\operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}) &= q^{\ell(w_{\mathbf{b}})} \prod_{t=1}^{p} \Big((-1)^d \varepsilon^{td} Q_1 \dots Q_d \Big)^{n-b_t} \\ &= (-1)^{dn(p-1)} q^{\ell(w_{\mathbf{b}})} \varepsilon^{\frac{1}{2}rn(p-1)-d\alpha(\mathbf{b})} (Q_1 \dots Q_d)^{n(p-1)},\end{aligned}$$

since $b_1 + \dots + b_p = n$. □

Remark 2.38. Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$. Then it is not difficult to see that

$$\ell(w_{\mathbf{b}}) = \sum_{1 \le i < j \le p} b_i b_j.$$

3. Specht modules and simple modules for $\mathscr{H}_{r,p,n}$

In this chapter we will introduce analogues of the Specht modules for $\mathscr{H}_{r,p,n}$ and hence construct a complete set of irreducible $\mathscr{H}_{r,p,n}$-modules. To do this we first use the results of the last chapter to prove Theorem B from the introduction. The first step is easy as Schur's Lemma easily implies that the element $z_{\mathbf{b}}$ acts on the Specht module $S(\boldsymbol{\lambda})$ of $\mathscr{H}_{r,n}$ as multiplication by a scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$ (Proposition 3.4). Using Theorems 2.31 and 2.35 we then compute $\mathfrak{f}_{\boldsymbol{\lambda}}$ explicitly in terms of the Schur elements of $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$ (Theorem 3.7). We show that the scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$ has a $p_{\boldsymbol{\lambda}}$th root and so complete the proof of Theorem B.

A calculation using seminormal forms and 'shifting homomorphisms' shows that, representation theoretically, taking the $p_{\boldsymbol{\lambda}}$th root of $\mathfrak{f}_{\boldsymbol{\lambda}}$ corresponds to the existence of an $\mathscr{H}_{r,p_{\boldsymbol{\lambda}},n}$-module endomorphism $\theta_{\boldsymbol{\lambda}}$ of $S(\boldsymbol{\lambda})$ such that $\theta_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}}$ is a scalar multiple of the identity map on $S(\boldsymbol{\lambda})$. As an $\mathscr{H}_{r,p,n}$-module, the Specht module $S(\boldsymbol{\lambda})$ then decomposes into a direct sum

$$S(\boldsymbol{\lambda}) = S_1^{\boldsymbol{\lambda}} \oplus S_2^{\boldsymbol{\lambda}} \oplus \dots \oplus S_{p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}}$$

of eigenspaces for $\theta_{\boldsymbol{\lambda}}$. The modules $S_t^{\boldsymbol{\lambda}}$, for $1 \le t \le p_{\boldsymbol{\lambda}}$, play the role of Specht modules for $\mathscr{H}_{r,p,n}$. Using some Clifford theory, we show in Theorem 3.50 that every irreducible $\mathscr{H}_{r,p,n}$-module arises as the simple head of some $S_t^{\boldsymbol{\lambda}}$ in a unique way, up to cyclic shift. These results complete the proof of Theorem C from the introduction.

Section 3.3 marks the real appearance of the Hecke algebras $\mathscr{H}_{r,p,n}$ of type $G(r,p,n)$ as, up until now, we have worked exclusively with $\mathscr{H}_{r,n}$-modules. In fact,

we do not really start working with $\mathscr{H}_{r,p,n}$ until section 3.7 where we construct Specht modules and simples modules for $\mathscr{H}_{r,p,n}$.

3.1. **Specht modules for $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$.** The algebras $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$ are both cellular algebras [11,20] with the cell modules of both algebras being called Specht modules. In this section we quickly recall the construction of these modules and the relationship between the Specht modules of these algebras.

First, recall that a **partition** of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of weakly decreasing non-negative integers which sum to $|\lambda| = n$. The **conjugate** of λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$, where $\lambda'_i = \#\{\, j \geq 1 \mid \lambda_j \geq i \,\}$.

An r**-multipartition** of n is an ordered r-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. Let $\mathscr{P}_{r,n}$ be the set of r-multipartitions of n. The partitions $\lambda^{(s)}$ are the **components** of $\boldsymbol{\lambda}$ and we call $\boldsymbol{\lambda}$ a multipartition when r is understood. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multipartition then its **conjugate** is the multipartition $\boldsymbol{\lambda}' = ({\lambda^{(r)}}', \dots, {\lambda^{(1)}}')$. To each multipartition $\boldsymbol{\lambda}$ we also associate a Young subgroup $\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ of $\mathfrak{S}_n$ in the obvious way.

The **diagram** of $\boldsymbol{\lambda}$ is the set $[\boldsymbol{\lambda}] = \{\, (i, j, s) \mid 1 \leq j \leq \lambda^{(s)}_i \text{ and } 1 \leq s \leq r \,\}$. A $\boldsymbol{\lambda}$**-tableau** is a map $\mathfrak{t} \colon [\boldsymbol{\lambda}] \longrightarrow \{1, 2, \dots, n\}$, which we think of as a labeling of the diagram of $\boldsymbol{\lambda}$. Thus we write $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ and we talk of the rows, columns and components of $\mathfrak{t}$. Let $\mathrm{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux.

By [11, Theorem 3.26], $\mathscr{H}_{r,n}$ is a cellular algebra with a cellular basis of the form

$$\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}), \text{ for } \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \,\}.$$

Hence, the cell modules of $\mathscr{H}_{r,n}$ are indexed by $\mathscr{P}_{r,n}$ and if $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ then the corresponding cell module $S(\boldsymbol{\lambda})$ has a basis of the form $\{\, m_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$.

Recall from the introduction that in $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ is a multipartition then $\boldsymbol{\lambda}^{[t]} = (\boldsymbol{\lambda}^{(dt-d+1)}, \boldsymbol{\lambda}^{(dt-d+2)}, \dots, \boldsymbol{\lambda}^{(dt)})$, for $1 \leq t \leq p$. More generally, set $\boldsymbol{\lambda}^{[t+kp]} = \boldsymbol{\lambda}^{[t]}$, for $k \in \mathbb{Z}$.

Definition 3.1. a) *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$. Then the **Specht module** $S(\boldsymbol{\lambda})$ for $\mathscr{H}_{r,n}$ is the cell module indexed by $\boldsymbol{\lambda}$ defined in [11, Definition 3.28].*

b) *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Then the **Specht module** for $\mathscr{H}_{d,\mathbf{b}}$ is the module $S_{\mathbf{b}}(\boldsymbol{\lambda}) \cong S(\boldsymbol{\lambda}^{[1]}) \otimes \cdots \otimes S(\boldsymbol{\lambda}^{[p]})$.*

We write $S^R(\boldsymbol{\lambda})$ when we want to emphasize that $S(\boldsymbol{\lambda})$ is an R-module. We will give an explicit construction of these modules in Section 3.4.

When $\mathscr{H}_{d,\mathbf{b}}$ is semisimple the modules $\{\, S_{\mathbf{b}}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}} \,\}$ give a complete set of pairwise non-isomorphic simple $\mathscr{H}_{d,\mathbf{b}}$-modules. Similarly, the modules $\{\, S(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \,\}$ give a complete set of pairwise non-isomorphic simple $\mathscr{H}_{r,n}$-modules when $\mathscr{H}_{r,n}$ is semisimple.

More generally, by the general theory of cellular algebras [20], each Specht module $S(\boldsymbol{\lambda})$ comes with an associative bilinear form and the radical $\operatorname{rad} S(\boldsymbol{\lambda})$ of this form is an $\mathscr{H}_{r,n}$-module. Define $D(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda}) / \operatorname{rad} S(\boldsymbol{\lambda})$. A multipartition $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ is **Kleshchev** if $D(\boldsymbol{\lambda}) \neq 0$. Let $\mathscr{K}_{r,n}(\mathbf{Q}^{\vee\varepsilon}) = \{\, \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \mid D(\boldsymbol{\lambda}) \neq 0 \,\}$ be the set of Kleshchev multipartitions of n. Then

$$\{\, D(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{K}_{r,n}(\mathbf{Q}^{\vee\varepsilon}) \,\}$$

is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{r,n}$-modules. Typically we write $\mathscr{K}_{r,n} = \mathscr{K}_{r,n}(\mathbf{Q}^{\vee\varepsilon})$ in what follows.

If A is an algebra and M is an A-module let $\operatorname{Head}(M)$ be the **head** of M. That is, M is the largest semisimple quotient of M. For example, by [11], if $\boldsymbol{\lambda}\in\mathscr{K}_{r,n}$ then $D(\boldsymbol{\lambda})=\operatorname{Head}(S(\boldsymbol{\lambda}))$. If S and D are modules for an algebra, with D irreducible, let $[S:D]$ be the multiplicity of D as a composition factor of S.

If $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are two multipartitions then $\boldsymbol{\lambda}$ **dominates** μ, and we write $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$ if

$$\sum_{s=1}^{t-1}|\boldsymbol{\lambda}^{(s)}|+\sum_{j=1}^{i}\boldsymbol{\lambda}^{(t)}_j\geq\sum_{s=1}^{t-1}|\boldsymbol{\mu}^{(s)}|+\sum_{j=1}^{i}\boldsymbol{\mu}^{(t)}_j$$

for $1\le t\le r$ and $i\ge 0$. We write $\boldsymbol{\lambda}\vartriangleright\boldsymbol{\mu}$ if $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$ and $\boldsymbol{\lambda}\neq\boldsymbol{\mu}$. The dominance partial order on $\mathscr{P}_{r,n}$ is useful because of the following fact.

Lemma 3.2 (([11, §3])). *Suppose that $[S(\boldsymbol{\lambda}):D(\boldsymbol{\mu})]\neq 0$, for $\boldsymbol{\lambda}\in\mathscr{P}_{r,n}$ and $\boldsymbol{\mu}\in\mathscr{K}_{r,n}$. Then $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$. Moreover, if $\boldsymbol{\mu}\in\mathscr{K}_{r,n}$ then $[S(\boldsymbol{\mu}):D(\boldsymbol{\mu})]=1$ and $D(\boldsymbol{\mu})=\operatorname{Head}S(\boldsymbol{\mu})$.*

Let $\mathscr{K}_{d,\mathbf{b}}=\{\,\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}\mid\boldsymbol{\lambda}^{[t]}\in\mathscr{K}_{d,b_t}(\varepsilon^t\mathbf{Q})\text{ for }1\le t\le p\,\}$. If $\boldsymbol{\lambda}\in\mathscr{K}_{d,\mathbf{b}}$ let

$$D_{\mathbf{b}}(\boldsymbol{\lambda})=S_{\mathbf{b}}(\boldsymbol{\lambda})/\operatorname{rad}S_{\mathbf{b}}(\boldsymbol{\lambda})\cong D(\boldsymbol{\lambda}^{[1]})\otimes\cdots\otimes D(\boldsymbol{\lambda}^{[p]}).$$

Theorem 2.5 and the remarks above imply that $\{\,D_{\mathbf{b}}(\boldsymbol{\lambda})\mid\boldsymbol{\lambda}\in\mathscr{K}_{d,\mathbf{b}}\,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{d,\mathbf{b}}$-modules.

Recall the functor $\mathtt{H}_{\mathbf{b}}$ from §2.5. By [12, Proposition 4.11] (see also [25, Proposition 2.13]), we have the following.

Lemma 3.3. *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$. Then*

a) $\mathtt{H}_{\mathbf{b}}\big(S_{\mathbf{b}}(\boldsymbol{\lambda})\big)\cong S(\boldsymbol{\lambda})$ *as $\mathscr{H}_{r,n}$-modules.*
b) $\mathtt{H}_{\mathbf{b}}\big(D_{\mathbf{b}}(\boldsymbol{\lambda})\big)\cong D(\boldsymbol{\lambda})$ *as $\mathscr{H}_{r,n}$-modules.*
c) $\boldsymbol{\lambda}=(\boldsymbol{\lambda}^{[1]},\dots,\boldsymbol{\lambda}^{[p]})\in\mathscr{K}_{d,\mathbf{b}}(\mathbf{Q}^{\vee\varepsilon})$ *is Kleshchev if and only if $\boldsymbol{\lambda}^{[t]}\in\mathscr{K}_{d,b_t}(\varepsilon^t\mathbf{Q})$, for $1\le t\le p$.*

In particular, we can consider $S(\boldsymbol{\lambda})\cong\mathtt{H}_{\mathbf{b}}\big(S_{\mathbf{b}}(\boldsymbol{\lambda})\big)=S_{\mathbf{b}}(\boldsymbol{\lambda})\cdot V_{\mathbf{b}}$ to be a submodule of $V_{\mathbf{b}}$.

3.2. **The scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$.** It follows from Schur's Lemma the central element $z_{\mathbf{b}}$ acts on the Specht modules $S(\boldsymbol{\lambda})$ as multiplication by a scalar, for $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$. In this section we explicitly compute this scalar, thus proving half of Theorem B from the introduction.

We define $\mathcal{A}=\mathbb{Z}[\dot\varepsilon,\dot q^{\pm1},\dot Q_1^{\pm1},\dots,\dot Q_d^{\pm1},A(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})^{-1}]$, where $\dot\varepsilon$ is a primitive pth root of unity in $\mathbb{C}$ and $\dot q$ and $\dot{\mathbf{Q}}=(\dot Q_1,\dots,\dot Q_d)$ are indeterminates over $\mathbb{Z}[\dot\varepsilon]$. Let $\mathcal{F}$ be the field of fractions of $\mathcal{A}$. If $\mathbf{Q}$ is (ε,q)-separated over R then R can be considered as an $\mathcal{A}$-module by letting $\dot\varepsilon$ act on R as multiplication by ε, $\dot q$ act as multiplication by q and $\dot Q_i$ act as multiplication by Q_i, for $1\le i\le d$. Therefore, $\mathscr{H}^R_{r,n}(q,\mathbf{Q})\cong\mathscr{H}^{\mathcal{A}}_{r,n}(\dot q,\dot{\mathbf{Q}})\otimes_{\mathcal{A}}R$ are isomorphic R-algebras. In particular, $\mathscr{H}^{\mathcal{F}}_{r,n}\cong\mathscr{H}^{\mathcal{A}}_{r,n}(\dot q,\dot{\mathbf{Q}})\otimes_{\mathcal{A}}\mathcal{F}$. The algebra $\mathscr{H}^{\mathcal{F}}_{r,n}$ is semisimple by Ariki's semisimplicity criteria [1]. The algebra $\mathscr{H}^{\mathcal{F}}_{r,n}$ is split semisimple because $\mathscr{H}^{\mathcal{F}}_{r,n}$ is a cellular algebra (and every field is a splitting field for a cellular algebra; see [20, Theorem 3.4]).

Abusing notation, we call the elements of $\mathcal{A}$ *polynomials* and if $f(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})\in\mathcal{A}$ then we define $f(\varepsilon,q,\mathbf{Q})=f(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})\cdot 1_R$ to be the value of $f(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})$ at $(\varepsilon,q,\mathbf{Q})$.

The scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$ in the next Proposition plays a key role in the proofs of all of our main results, Theorems A–D, from the introduction.

Proposition 3.4. *Suppose that* $\mathbf{Q}$ *is* (ε,q)*-separated in* R *and that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and* $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$. *Then there exists a non-zero scalar* $\mathfrak{f}_{\boldsymbol{\lambda}}\in R$ *such that*

$$z_{\mathbf{b}}\cdot x=\mathfrak{f}_{\boldsymbol{\lambda}}x,$$

for all $x\in S(\boldsymbol{\lambda})$. *Moreover, there exists a non-zero polynomial* $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}=\mathfrak{f}_{\boldsymbol{\lambda}}(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})\in\mathcal{A}$ *such that* $\mathfrak{f}_{\boldsymbol{\lambda}}=\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}(\varepsilon,q,\mathbf{Q})\in R$.

Proof. The Specht module $S_{\mathbf{b}}(\boldsymbol{\lambda})$ is free as an R-module so, by the remarks above, $S_{\mathbf{b}}(\boldsymbol{\lambda})\cong S^{\mathcal{A}}_{\mathbf{b}}(\boldsymbol{\lambda})\otimes_{\mathcal{A}}R$. Therefore, to show that such a scalar exists it is enough to consider the case when $R=\mathcal{A}$. Similarly, since $S^{\mathcal{A}}_{\mathbf{b}}(\boldsymbol{\lambda})$ embeds into $S^{\mathcal{F}}_{\mathbf{b}}(\boldsymbol{\lambda})\cong S^{\mathcal{A}}_{\mathbf{b}}(\boldsymbol{\lambda})\otimes_{\mathcal{A}}\mathcal{F}$ we may assume that $R=\mathcal{F}$. By the remarks above, the algebra $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ is split semisimple and the module $S^{\mathcal{F}}_{\mathbf{b}}(\boldsymbol{\lambda})$ is an irreducible $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$-module, so by Schur's Lemma the homomorphism of $S_{\mathbf{b}}(\boldsymbol{\lambda})$ given by left multiplication by $z_{\mathbf{b}}$ is equal to multiplication by some scalar $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$. Notice that $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ is an element of $\mathcal{A}$ because $z_{\mathbf{b}}\cdot v_{\mathbf{b}}T_{\mathbf{b}}\in\mathscr{H}^{\mathcal{A}}_{r,n}$. By specialization, the scalar $\mathfrak{f}_{\boldsymbol{\lambda}}\in R$ in the statement of the Lemma is given by evaluating the polynomial $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})$ at $(\varepsilon,q,\mathbf{Q})$. Finally, observe that $\mathfrak{f}_{\boldsymbol{\lambda}}\neq 0$ since $z_{\mathbf{b}}$ acts invertibly on $V_{\mathbf{b}}$ by Lemma 2.28. □

We will determine the scalar $\mathfrak{f}_{\boldsymbol{\lambda}}\in R$ by computing the polynomial $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ in $\mathcal{A}$. In fact, we have already done all of the work needed to determine $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$. To describe $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ we only need one definition.

Abusing notation slightly, let Tr be the trace form on $\mathscr{H}^{\mathcal{F}}_{r,n}$ given by (2.33). Let $\chi^{\boldsymbol{\lambda}}$ be the character of $S^{\mathcal{F}}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda}\in\mathscr{P}_{r,n}$. Then $\{\,\chi^{\boldsymbol{\lambda}}\mid\boldsymbol{\lambda}\in\mathscr{P}_{r,n}\,\}$ is a complete set of pairwise inequivalent irreducible characters for $\mathscr{H}^{\mathcal{F}}_{r,n}$, so it is a basis for the space of trace functions on $\mathscr{H}^{\mathcal{F}}_{r,n}$. In particular, Tr can be written in a unique way as a linear combination of the irreducible characters. Moreover, it is easy to see that every character $\chi^{\boldsymbol{\lambda}}$ must appear in Tr with *non-zero coefficient* because Tr is non-degenerate; see, for example, [16, Example 7.1.3]. Consequently, the following definition makes sense.

Definition 3.5. *The* ***Schur elements*** *of* $\mathscr{H}^{\mathcal{F}}_{r,n}$ *are the scalars* $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}=\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})\in\mathcal{F}$, *for* $\boldsymbol{\lambda}\in\mathscr{P}_{r,n}$, *such that*

$$\mathrm{Tr}=\sum_{\boldsymbol{\lambda}\in\mathscr{P}_{r,n}}\frac{1}{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}}\chi^{\boldsymbol{\lambda}}.$$

For $\boldsymbol{\lambda}\in\mathscr{P}_{r,n}$ fix $F_{\boldsymbol{\lambda}}$ a primitive idempotent in $\mathscr{H}^{\mathcal{F}}_{r,n}$ such that $F_{\boldsymbol{\lambda}}\mathscr{H}^{\mathcal{F}}_{r,n}\cong S^{\mathcal{F}}(\boldsymbol{\lambda})$. Using, for example seminormal forms $\mathscr{H}^{\mathcal{F}}_{r,n}$ [28, Theorem 2.11], it is easy to see that $\chi^{\boldsymbol{\lambda}}(F_{\boldsymbol{\mu}})=\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}$, for $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{r,n}$. Hence, a second characterisation of the Schur elements is that

$$\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}=\frac{1}{\mathrm{Tr}(F_{\boldsymbol{\lambda}})}.\tag{3.6}$$

Similarly, for each $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ the trace form $\mathrm{Tr}_{\mathbf{b}}$ determines Schur elements $\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}\in\mathcal{F}$ for $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$. By the remarks above, the Schur elements of $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ satisfy

$$\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}=\prod_{t=1}^{p}\dot{\mathfrak{s}}_{\boldsymbol{\lambda}^{[t]}}(\dot\varepsilon,\dot q,\dot\varepsilon^t\dot{\mathbf{Q}})=\frac{1}{\mathrm{Tr}_{\mathbf{b}}(F_{\mathbf{b}}(\boldsymbol{\lambda}))},$$

where $F_{\mathbf{b}}(\boldsymbol{\lambda})$ is a primitive idempotent in $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ such that $S^{\mathcal{F}}_{\mathbf{b}}(\boldsymbol{\lambda})\cong F_{\mathbf{b}}(\boldsymbol{\lambda})\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$.

Theorem 3.7. *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$ *and that* $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$*. Then*

$$\dot{\mathfrak{f}}_{\boldsymbol{\lambda}} = \frac{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}}{\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}} \operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}).$$

Consequently, $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}} = (-1)^{n(r-d)} \dot{q}^{\ell(w_{\mathbf{b}})} \dot{\varepsilon}^{\frac{1}{2}rn(p-1)-d\alpha(\mathbf{b})} (\dot{Q}_1 \dots \dot{Q}_d)^{n(p-1)} \dfrac{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}}{\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}}$.

Proof. To compute $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ we may assume that $R = \mathcal{F}$ and work in $\mathscr{H}^{\mathcal{F}}_{r,n}$. Let $F_{\mathbf{b}}(\boldsymbol{\lambda})$ be a primitive idempotent in $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ such that $S^{\mathcal{F}}_{\mathbf{b}}(\boldsymbol{\lambda}) \cong F_{\mathbf{b}}(\boldsymbol{\lambda})\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$. Then $F_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot e_{\mathbf{b}}$ is a primitive idempotent in $\mathscr{H}^{\mathcal{F}}_{r,n}$ such that $F_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot e_{\mathbf{b}} \mathscr{H}^{\mathcal{F}}_{r,n} \cong S^{\mathcal{F}}(\boldsymbol{\lambda})$ by Theorem 2.31 and Lemma 3.3. Therefore, using the remarks above,

$$\begin{aligned}
\frac{1}{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}} &= \operatorname{Tr}(F_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot e_{\mathbf{b}}) = \operatorname{Tr}(z_{\mathbf{b}}^{-1} F_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot v_{\mathbf{b}} T_{\mathbf{b}}), && \text{since } z_{\mathbf{b}} \text{ is central in } \mathscr{H}_{d,\mathbf{b}},\\
&= \frac{1}{\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}} \operatorname{Tr}(F_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot v_{\mathbf{b}} T_{\mathbf{b}}), && \text{by Proposition 3.4},\\
&= \frac{1}{\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}} \operatorname{Tr}_{\mathbf{b}}(F_{\mathbf{b}}(\boldsymbol{\lambda})) \operatorname{Tr}(\mathfrak{v}_{\mathbf{b}} T_{\mathbf{b}}), && \text{by Theorem 2.35},\\
&= \frac{1}{\dot{\mathfrak{f}}_{\boldsymbol{\lambda}} \dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}} \operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}).
\end{aligned}$$

Rearranging this equation gives the first formula for $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$. Applying Corollary 2.37 proves the second. □

Remark 3.8. The proof of Theorem 3.7 is deceptively easy: all of the hard work is done in proving Theorem 2.31 and Theorem 2.35.

We want to make the formula for $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ more explicit. To do this we recall the elegant closed formula for the Schur elements obtained by Chlouveraki and Jacon [8]. Before we can state their result we need some notation. First, for $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ define $\overrightarrow{\boldsymbol{\lambda}}$ to be the partition obtained from $\boldsymbol{\lambda}$ by putting all of the parts of $\boldsymbol{\lambda}$ in weakly decreasing order. (For example, if $\boldsymbol{\lambda} = \big((2,1^2),(3,2,1)\big)$ then $\overrightarrow{\boldsymbol{\lambda}} = (3,2^2,1^3)$.) Next, if λ is a partition define

$$\beta(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i = \sum_{i \geq 1} \binom{\lambda'_i}{2},$$

where $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is the partition conjugate to λ (the second equality is well-known and straightforward to check). Given two partitions λ and μ and $(i,j) \in [\lambda]$ define

$$h_{ij}(\lambda,\mu) = \lambda_i - i + \mu'_j - j + 1,$$

which Chlouveraki and Jacon call a generalised **hook length**. Observe that if $1 \leq s \leq r$ then s can be written uniquely in the form $s = d(p_s - 1) + d_s$, where $1 \leq p_s \leq p$ and $1 \leq d_s \leq d$. Then, as in (1.3), $\lambda^{(s)}$ is the d_sth component of $\boldsymbol{\lambda}^{[p_s]}$. Finally, if $(i,j,s) \in [\boldsymbol{\lambda}]$ and $1 \leq t \leq r$ then set

$$h^{\boldsymbol{\lambda}}_{ij}(s,t) = \dot{\varepsilon}^{p_s - p_t} \dot{q}^{h_{ij}(\lambda^{(s)},\lambda^{(t)})} \dot{Q}_{d_s} \dot{Q}^{-1}_{d_t}.$$

After translating their notation to our setting, Chlouveraki and Jacon [8, Theorem 3.2] show that

$$\dot{\mathfrak{s}}_{\boldsymbol{\lambda}} = (-1)^{n(r-1)} \dot{q}^{-\beta(\overrightarrow{\boldsymbol{\lambda}})} (\dot{q}-1)^{-n} \prod_{(i,j,s) \in [\boldsymbol{\lambda}]} \prod_{1 \leq t \leq r} (h^{\boldsymbol{\lambda}}_{ij}(s,t) - 1).$$

There is an analogous formula for $\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol\lambda} = \prod_{t=1}^{p} \dot{\mathfrak{s}}_{\boldsymbol\lambda^{[t]}}$ which can be obtained by setting $p=1$. Using Theorem 3.7, and the equations above, we obtain the following closed formula for $\dot{\mathfrak{f}}_{\boldsymbol\lambda}$.

Corollary 3.9. *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$ *and that* $\boldsymbol\lambda \in \mathscr{P}_{d,\mathbf{b}}$. *Then*

$$\dot{\mathfrak{f}}_{\boldsymbol\lambda} = \dot\varepsilon^{\frac12 rn(p-1)-d\alpha(\mathbf{b})}\dot q^{\gamma_{\mathbf{b}}(\boldsymbol\lambda)}(\dot Q_1\dots\dot Q_d)^{n(p-1)}\prod_{(i,j,s)\in[\boldsymbol\lambda]}\prod_{\substack{1\le t\le r\\ p_t\ne p_s}}(h^{\boldsymbol\lambda}_{ij}(s,t)-1),$$

where $\gamma_{\mathbf{b}}(\boldsymbol\lambda) = \ell(w_{\mathbf{b}}) - \beta(\overrightarrow{\boldsymbol\lambda}) + \sum_{a=1}^{p}\beta(\overrightarrow{\boldsymbol\lambda^{[a]}})$. *In particular,* $\dot{\mathfrak{f}}_{\boldsymbol\lambda} \in \mathbb{Z}[\dot\varepsilon, \dot q^{\pm1}, \dot Q_1^{\pm1},\dots,\dot Q_d^{\pm1}]$.

If $\boldsymbol\lambda \in \mathscr{P}_{r,n}$ then $\dot{\mathfrak{f}}_{\boldsymbol\lambda} \in \mathcal{A}$ by Proposition 3.4. Corollary 3.9 establishes the stronger result that $\dot{\mathfrak{f}}_{\boldsymbol\lambda}$ is a Laurent polynomial in $\mathbb{Z}[\dot\varepsilon, \dot q^{\pm1}, \dot Q_1^{\pm1},\dots,\dot Q_d^{\pm1}]$. Using the definitions it is easy to see that if $(i,j,s)\in[\boldsymbol\lambda]$ and $1\le t\le r$ then $h^{\boldsymbol\lambda}_{ij}(s,t)-1$ divides $A(\varepsilon,q,\mathbf{Q})$. Therefore, it is self-evident from Corollary 3.9 that $\mathfrak{f}_{\boldsymbol\lambda} = \dot{\mathfrak{f}}_{\boldsymbol\lambda}(\varepsilon,q,\mathbf{Q})$ is both well-defined and non-zero whenever $\mathbf{Q}$ is (ε,q)-separated over R.

3.3. Graded Clifford systems.

We will use Clifford theory extensively in order to understand the representation theory of $\mathscr{H}_{r,p,n}$ in terms of the representation theory of $\mathscr{H}_{r,n}$. This section recalls the theory that we need most and starts applying it to the algebras $\mathscr{H}_{r,p,n}$.

Suppose that A is a finitely generated R-algebra. A family of R-submodules $\{\, A_s \mid s\in\mathbb{Z}/p\mathbb{Z}\,\}$ is a $\mathbb{Z}/p\mathbb{Z}$**-graded Clifford** system if the following conditions are satisfied:

a) $A_sA_t = A_{s+t}$ for any $s,t\in\mathbb{Z}/p\mathbb{Z}$;
b) For each $s\in\mathbb{Z}/p\mathbb{Z}$, there is a unit $a_s\in A_s$ such that $A_s = a_sA_0 = A_0a_s$;
c) $A = \oplus_{s\in\mathbb{Z}/p\mathbb{Z}}A_s$;
d) $1\in A_0$.

Any automorphism α of an R-algebra A induces an equivalence $\mathtt{F}^\alpha\colon$ Mod-$A\longrightarrow$ Mod-A. Explicitly, if M is an A-module then $\mathtt{F}^\alpha(M) = M^\alpha$ is the A-module which is equal to M as an R-module but with the action *twisted* by α so that if $m\in M$ and $x\in A$ then $m\cdot x = mx^\alpha = m\alpha(x)$, where on the right hand side we have the usual (untwisted) action of A.

The following general result is proved in [17, Proposition 2.2], together with [23, Appendix] which corrects a gap in the original argument. Recall that we have assumed that R contains a primitive pth root of unity ε.

Lemma 3.10. *Suppose that* A *and* B *finitely generated* R*-free* R*-algebras such that* $A = \bigoplus_{t=0}^{p-1} B\theta^t$ *where* θ *is a unit in* A *such that* $\theta^p\in B$ *and* $\theta B = B\theta$. *Then there is an isomorphism of* (A,A)*-bimodules*

$$A\otimes_B A \cong \bigoplus_{t=0}^{p-1} A^{\theta^t};\, b\theta^i\otimes\theta^j \mapsto \sum_{t=0}^{p-1}(\varepsilon^{jt}b\theta^{i+j})_{(t)},$$

for $b\in B$ *and* $0\le i,j<p$ *and where* $(\varepsilon^{jt}b\theta^{i+j})_{(t)}\in A^{\theta^t}$. *Here we view* $\bigoplus_t A^{\theta^t}$ *as an* (A,A)*-bimodule by making* A *act from the left as left multiplication and from the right on* A^{θ^t} *as right multiplication twisted by* θ^t, *for* $0\le t<p$.

An explicit isomorphism, as in the lemma, is constructed in [23, p. 3391].

In the setup of Lemma 3.10 the subspaces $\{\, B\theta^s \mid s \in \mathbb{Z}/p\mathbb{Z} \,\}$ form a $\mathbb{Z}/p\mathbb{Z}$-graded Clifford system in A. Now we assume that $R = K$ is a field. Let α be the automorphism of B given by $\alpha(b) = \theta b\theta^{-1}$, for $b \in B$. Let β be the automorphism of A given by $\beta(b\theta^j) = \varepsilon^j b\theta^j$, for $b \in B$ and $j \in \mathbb{Z}/p\mathbb{Z}$.

Let $\operatorname{Irr}(A)$ and $\operatorname{Irr}(B)$ be the sets of isomorphism classes of simple A-modules and simple B-modules, respectively. For each $D(\lambda) \in \operatorname{Irr}(A)$ fix a simple B-submodule D^λ of $D(\lambda)\downarrow_B^A$. It is clear that $D(\lambda)^\alpha \cong D(\lambda)$ and $(D^\lambda)^\beta \cong D^\lambda$. Let o_λ be the smallest positive integer such that $D(\lambda)^{\beta^{\mathsf{o}_\lambda}} \cong D(\lambda)$. Then o_λ divides p so we set $p_\lambda = p/\mathsf{o}_\lambda$. Define an equivalence relation $\sim_\beta$ on $\operatorname{Irr}(A)$ by declaring that

$$D(\lambda) \sim_\beta D(\mu) \Longleftrightarrow D(\lambda) \cong D(\mu)^{\beta^t}, \quad \text{for some } t \in \mathbb{Z}/p\mathbb{Z}.$$

Similarly, let $\sim_\alpha$ be the equivalence relation on $\operatorname{Irr}(B)$ given by

$$D^\lambda \sim_\alpha D^\mu \Longleftrightarrow D^\lambda \cong (D^\mu)^{\alpha^t}, \quad \text{for some } t \in \mathbb{Z}/p\mathbb{Z}.$$

If D is an A-module let $\operatorname{Soc}_A(M)$ be its **socle**; that is the maximal semisimple submodule of A. Similarly, recall that $\operatorname{Head}_A(M)$ is the maximal semisimple quotient of M.

The following result is similar to [18, Lemma 2.2]. The result in [18] is proved only in the case $R = \mathbb{C}$. As we now show, the argument applies over any algebraically closed field.

Lemma 3.11 ((cf. [18, Lemma 2.2])). *Suppose that $R = K$ is an algebraically closed field and that $A = \bigoplus_{t=0}^{p-1} B\theta^t$ as in Lemma 3.10.*

a) *Suppose that $D(\lambda) \in \operatorname{Irr}(A)$. Then p_λ is the smallest positive integer such that $D^\lambda \cong \big(D^\lambda\big)^{\alpha^{p_\lambda}}$.*
b) *Suppose that $D^\lambda \in \operatorname{Irr}(B)$. Then $D^\lambda\uparrow_B^A \cong D(\lambda) \oplus D(\lambda)^\beta \oplus \cdots \oplus D(\lambda)^{\beta^{\mathsf{o}_\lambda - 1}}$ and $D(\lambda)\downarrow_B^A \cong D^\lambda \oplus (D^\lambda)^\alpha \oplus \cdots \oplus \big(D^\lambda\big)^{\alpha^{(p_\lambda - 1)}}$.*
c) *$\{\, (D^\lambda)^{\alpha^i} \mid D(\lambda) \in \operatorname{Irr}(A)/{\sim_\beta}$ for $1 \le i \le p_\lambda \,\}$ is a complete set of pairwise non-isomorphic absolutely irreducible B-modules.*
d) *$\{\, D(\lambda)^{\beta^i} \mid D^\lambda \in \operatorname{Irr}(B)/{\sim_\alpha}$ for $1 \le i \le o_\lambda \,\}$ is a complete set of pairwise non-isomorphic absolutely irreducible A-modules.*

Proof. Let $D(\lambda) \in \operatorname{Irr}(A)$. Let p'_λ be the smallest positive integer such that $D^\lambda \cong \big(D^\lambda\big)^{\alpha^{p'_\lambda}}$. By [9, Proposition 11.16], the module $D(\lambda)\downarrow_B^A$ is semisimple. Now,

$$\operatorname{Hom}_A\big(D^\lambda, D(\lambda)\downarrow_B^A\big) \cong \operatorname{Hom}_A\big((D^\lambda)^{\alpha^t}, D(\lambda)\downarrow_B^A\big), \quad \text{for any } t \in \mathbb{Z}.$$

Therefore, there exists an integer $c > 0$ such that

$$D(\lambda)\downarrow_B^A \cong \big(D^\lambda \oplus (D^\lambda)^\alpha \oplus \cdots \oplus (D^\lambda)^{\alpha^{p'_\lambda - 1}}\big)^{\oplus c}. \tag{3.12}$$

By Frobenius Reciprocity,

$$\operatorname{Hom}_B\big(D(\lambda)\downarrow_B^A, D^\lambda\big) \cong \operatorname{Hom}_A\big(D(\lambda), D^\lambda\uparrow_B^A\big).$$

Since K is algebraically closed, both A and B are split over K. It follows that

$$\big(D(\lambda) \oplus D(\lambda)^\beta \oplus \cdots \oplus D(\lambda)^{\beta^{\mathsf{o}_\lambda - 1}}\big)^{\oplus c} \subseteq \operatorname{Soc}_A\big(D^\lambda\uparrow_B^A\big). \tag{3.13}$$

By (3.12) and (3.13), we have that

$$\dim D(\lambda) = cp'_\lambda \dim D^\lambda \quad \text{and} \quad p \dim D^\lambda \ge c\mathsf{o}_\lambda \dim D(\lambda). \tag{3.14}$$

Hence,

$$p \geq c^2 p'_\lambda \mathsf{o}_\lambda. \tag{3.15}$$

On the other hand, since R contains a primitive pth root of unity, the integer p and all of its divisors are invertible in R. Let π_λ be a linear endomorphism of $D(\lambda)$ which induces an A-module isomorphism $D(\lambda) \cong D(\lambda)^{\beta^{o_\lambda}}$. Then $(\pi_\lambda)^{p_\lambda} \in \mathrm{End}_A\big(D(\lambda)\big) = K$. Renormalising π_λ, if necessary, we can assume that $(\pi_\lambda)^{p_\lambda} = \mathrm{id}_\lambda$, where id_λ is the identity map on $D(\lambda)$.

Let X be an indeterminate over K and suppose then o divides p. Differentiating the identity $X^{p/\mathsf{o}} - 1 = \prod_{j=1}^{p/\mathsf{o}}(X - \varepsilon^{j\mathsf{o}})$ and setting $X = \pi_\lambda$ and $\mathsf{o} = \mathsf{o}_\lambda$, shows that

$$p_\lambda \pi_\lambda^{p_\lambda - 1} = \sum_{j=1}^{p_\lambda} \prod_{\substack{1 \leq t \leq p_\lambda \\ t \neq j}} (\pi_\lambda - \varepsilon^{t\mathsf{o}_\lambda}).$$

Thus,

$$\mathrm{id}_\lambda = \frac{1}{p_\lambda} \sum_{j=1}^{p_\lambda} \prod_{\substack{1 \leq t \leq p_\lambda \\ t \neq j}} (\pi_\lambda - \varepsilon^{t\mathsf{o}_\lambda}) \pi_\lambda^{1-p_\lambda}.$$

For each integer $1 \leq j \leq p_{\boldsymbol{\lambda}}$, we define

$$D_j(\lambda) := \frac{1}{p_\lambda} \prod_{\substack{1 \leq t \leq p_\lambda \\ t \neq j}} (\pi_\lambda - \varepsilon^{t\mathsf{o}_\lambda})\, \pi_\lambda^{1-p_\lambda}\, D(\lambda).$$

It is easy to check that each $D_j(\lambda)$ is a B-submodule of $D(\lambda)\!\downarrow_B^A$ and $D_j(\lambda)\theta = D_{j+1}(\lambda)$ for each $j \in \mathbb{Z}/p\mathbb{Z}$. In particular, this implies that $D(\lambda)\!\downarrow_B^A$ can be decomposed into a direct sum of $p_{\boldsymbol{\lambda}}$ nonzero B-submodules. Comparing this with (3.12), we deduce that $p_{\boldsymbol{\lambda}} = p/\mathsf{o}_{\boldsymbol{\lambda}} \leq c p'_\lambda$. Combining this with (3.15), shows that $c^2 p'_\lambda \mathsf{o}_\lambda \leq p \leq c p'_\lambda \mathsf{o}_\lambda$, which forces that $c = 1$, $p = \mathsf{o}_\lambda p'_\lambda$, and

$$D^\lambda\!\uparrow_B^A = \mathrm{Soc}_A\big(D^\lambda\!\uparrow_B^A\big) = D(\lambda) \oplus D(\lambda)^\beta \oplus \cdots \oplus D(\lambda)^{\beta^{\mathsf{o}_\lambda - 1}}.$$

This proves the first two statements of the lemma. The last two statements follow by Frobenius reciprocity using the first two statements. □

We now apply these results to $\mathscr{H}_{r,p,n}$. Recall from Section 2.1 that $\mathscr{H}_{r,n}$ has two automorphisms σ and τ such that $\mathscr{H}_{r,p,n}$ is the σ-fixed point subalgebra of $\mathscr{H}_{r,n}$ and τ restricts to an automorphism of $\mathscr{H}_{r,p,n}$.

It is straightforward to check that, as a right $\mathscr{H}_{r,p,n}$-module,

$$\mathscr{H}_{r,n} = \mathscr{H}_{r,p,n} \oplus T_0 \mathscr{H}_{r,p,n} \oplus \cdots \oplus T_o^{p-1} \mathscr{H}_{r,p,n}.$$

(For example, use [25, Lemma 3.1].) Hence, $\mathscr{H}_{r,n}$ is a $\mathbb{Z}/p\mathbb{Z}$-graded Clifford system over $\mathscr{H}_{r,p,n}$. Applying Lemma 3.10 to $\mathscr{H}_{r,n} = \bigoplus_{i=0}^{p-1} \mathscr{H}_{r,p,n} T_0^i$ we obtain the following useful result.

Proposition 3.16. *There is a natural isomorphism of $(\mathscr{H}_{r,n}, \mathscr{H}_{r,n})$-bimodules*

$$\mathscr{H}_{r,n} \otimes_{\mathscr{H}_{r,p,n}} \mathscr{H}_{r,n} \cong \bigoplus_{m=0}^{p-1} \big(\mathscr{H}_{r,n}\big)^{\sigma^m},$$

where $\mathscr{H}_{r,n}$ acts from the left on $\big(\mathscr{H}_{r,n}\big)^{\sigma^m}$ as left multiplication and from the right with its action twisted by σ^m.

Corollary 3.17. *Suppose that M is an $\mathscr{H}_{r,n}$-module. Then, as $\mathscr{H}_{r,n}$-modules,*

$$M\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\cong\bigoplus_{i=0}^{p-1}M^{\sigma^i}.$$

Proof. By definition, $M\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}=M\otimes_{\mathscr{H}_{r,n}}\mathscr{H}_{r,n}\otimes_{\mathscr{H}_{r,p,n}}\mathscr{H}_{r,n}$. Now apply Proposition 3.16. □

3.4. **Twisting modules by σ.** In this section we investigate the effect on $\mathscr{H}_{r,n}$-modules of twisting by the automorphism σ defined in Section 2.1. These results will be useful when showing that $\mathfrak{f}_{\boldsymbol{\lambda}}$ has a $p_{\boldsymbol{\lambda}}$ root and when constructing and classifying the irreducible $\mathscr{H}_{r,p,n}$-modules.

Recall that $\sigma(T_0)=\varepsilon T_0$ and that $\sigma(T_i)=T_i$, for $1\le i<n$. It is easy to check that $\sigma(T_w)=T_w$ and that $\sigma(L_m)=\varepsilon L_m$, for $w\in\mathfrak{S}_n$ and for $1\le m\le n$. Hence, using the definitions we obtain the following.

Lemma 3.18. *Suppose that $1\le b\le n$ and $1\le s\le t\le p$. Then*

$$\sigma(\mathcal{L}_{1,b}^{(s,t)})=\varepsilon^{bd(t-s+1)}\mathcal{L}_{1,b}^{(s-1,t-1)}.$$

Consequently, if $\mathbf{b}\in\mathscr{C}_{p,n}$ then $\sigma(v_{\mathbf{b}})=\varepsilon^{-nd}v_{\mathbf{b}}^{(-1)}$ and $\sigma(Y_t)=\varepsilon^{-db_t}Y_{t-1}$, for $1\le t\le p$.

By the remarks in Section 3.3, the automorphism σ induces a functor $\mathtt{F}^\sigma$ on the category of $\mathscr{H}_{r,n}$-modules. We want to compare $\mathtt{F}^\sigma$ with the functors $\mathtt{H}_{\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$, which appear in the Morita equivalences of Theorem 2.5.

Lemma 3.19. *Let $\mathbf{b}\in\mathscr{C}_{p,n}$ and $t\in\mathbb{Z}$. Suppose that $\mathbf{Q}$ is (ε,q)-separated over K. Then $V_{\mathbf{b}\langle t\rangle}\cong V^\sigma_{\mathbf{b}\langle t+1\rangle}$.*

Proof. Let $\varsigma=\sigma^{-1}$. Then it is enough to show that $V_{\mathbf{b}}^\varsigma\cong V_{\mathbf{b}\langle 1\rangle}$ which is equivalent to the statement in the Lemma when $t=0$. By Corollary 2.29, there is an isomorphism $V_{\mathbf{b}}\xrightarrow{\simeq}V^{(1)}_{\mathbf{b}\langle 1\rangle}$. On the other hand, $V^\varsigma_{\mathbf{b}}\cong\sigma(V_{\mathbf{b}})\cong V_{\mathbf{b}}^{(-1)}$ by Lemma 3.18. Therefore, the map $v\mapsto(Y_1v)^\varsigma$, for $v\in V_{\mathbf{b}}$, gives the required isomorphism $V^\varsigma_{\mathbf{b}}\xrightarrow{\simeq}V_{\mathbf{b}\langle 1\rangle}$. □

Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$ and recall that, by definition,

$$\mathscr{H}_{d,\mathbf{b}}=\mathscr{H}_{d,\mathbf{b}}(\mathbf{Q}^{\vee\varepsilon})=\mathscr{H}_{d,b_1}(\varepsilon\mathbf{Q})\otimes\cdots\otimes\mathscr{H}_{d,b_p}(\varepsilon^p\mathbf{Q}).$$

Suppose that $h=h_1\otimes\cdots\otimes h_p\in\mathscr{H}_{d,\mathbf{b}}$. Applying the relations, there is an algebra isomorphism

$$\mathscr{H}_{d,\mathbf{b}}\xrightarrow{\simeq}\mathscr{H}_{d,\mathbf{b}\langle -1\rangle};h_1\otimes\cdots\otimes h_p\mapsto h\langle -1\rangle=h_p^\sigma\otimes h_1^\sigma\otimes\cdots\otimes h_{p-1}^\sigma,\tag{3.20}$$

where we abuse notation slightly and define $\sigma(T_0^{(t)})=\varepsilon^{-1}T_0^{(t+1)}$ and $\sigma(T_i^{(t)})=T_i^{(t+1)}$, for $1\le i<b_t$ and where we equate superscripts modulo p. It follows that there is an equivalence of categories $\mathtt{F}^\sigma_{\mathbf{b}}\colon\text{Mod-}\mathscr{H}_{d,\mathbf{b}}\longrightarrow\text{Mod-}\mathscr{H}_{d,\mathbf{b}\langle -1\rangle}$ given by

$$\mathtt{F}^\sigma_{\mathbf{b}}(M_1\otimes\cdots\otimes M_p)=M_p\otimes M_1\otimes\cdots\otimes M_{p-1},$$

for an $\mathscr{H}_{d,\mathbf{b}}$-module $M_1\otimes\cdots\otimes M_p$ and where $\mathscr{H}_{d,\mathbf{b}\langle -1\rangle}$ acts via the isomorphism above.

Proposition 3.21. *Let $\mathbf{b} \in \mathscr{C}_{p,n}$. Suppose that $\mathbf{Q}$ is (ε, q)-separated over K. Then*

$$\begin{array}{ccc} \text{Mod-}\mathscr{H}_{d,\mathbf{b}} & \xrightarrow{\mathtt{F}^\sigma_{\mathbf{b}}} & \text{Mod-}\mathscr{H}_{d,\mathbf{b}\langle -1\rangle} \\ {\scriptstyle \mathtt{H}_{\mathbf{b}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathtt{H}_{\mathbf{b}\langle -1\rangle}} \\ \text{Mod-}\mathscr{H}_{r,n} & \xrightarrow[\mathtt{F}^\sigma]{} & \text{Mod-}\mathscr{H}_{r,n} \end{array}$$

is a commutative diagram of functors.

Proof. Let M be an $\mathscr{H}_{d,\mathbf{b}}$-module. Then we have to prove that

$$\big(M \otimes_{\mathscr{H}_{d,\mathbf{b}}} V_{\mathbf{b}}\big)^\sigma \cong \mathtt{F}^\sigma_{\mathbf{b}}(M) \otimes_{\mathscr{H}_{d,\mathbf{b}\langle -1\rangle}} V_{\mathbf{b}\langle -1\rangle}$$

as right $\mathscr{H}_{r,n}$-modules. Mimicking the proof of Lemma 3.19, the required isomorphism is the map $m \otimes v \mapsto m\langle -1\rangle \otimes (Y_1 v)^\sigma$, for $m \otimes v \in M \otimes_{\mathscr{H}_{d,\mathbf{b}}} V_{\mathbf{b}}$. □

We want to use this result to determine the σ-twists of various $\mathscr{H}_{r,n}$-modules. To this end set $a^{\boldsymbol{\lambda}}_{s,t} = |\lambda^{(dt-d+1)}| + \cdots + |\lambda^{(dt-d+s-1)}|$, for $1 \le i \le d$ and $1 \le t \le p$ and define

$$u^+_{\boldsymbol{\lambda}^{[t]}} = u^+_{\boldsymbol{\lambda}^{[t]}}(\varepsilon^t\mathbf{Q}) = \prod_{s=2}^{d}\prod_{j=1}^{a^{\boldsymbol{\lambda}}_{s,t}}(L_j - \varepsilon^t Q_s) \qquad \text{and} \qquad x_{\boldsymbol{\lambda}^{[t]}} = \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}^{[t]}}} T_w,$$

$$y_{\boldsymbol{\lambda}^{[t]}} = \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}^{[t]}}} (-1)^{\ell(w)} T_w,$$

which we think of as elements of $\mathscr{H}_{d,b_t}(\varepsilon^t\mathbf{Q})$ in the natural way. Now set $u^+_{\boldsymbol{\lambda},\mathbf{b}} = u^+_{\boldsymbol{\lambda}^{[1]}} \otimes \cdots \otimes u^+_{\boldsymbol{\lambda}^{[p]}}$ and $x_{\boldsymbol{\lambda},\mathbf{b}} = x_{\boldsymbol{\lambda}^{[1]}} \otimes \cdots \otimes x_{\boldsymbol{\lambda}^{[p]}}$. We remark that it is easy to check that $u^+_{\boldsymbol{\lambda},\mathbf{b}}$ and $x_{\boldsymbol{\lambda},\mathbf{b}}$ commute using Lemma 2.8.

By [14, Theorem 2.9], there exists an element $s_{\mathbf{b}}(\boldsymbol{\lambda}) = s(\boldsymbol{\lambda}^{[1]}) \otimes \cdots \otimes s(\boldsymbol{\lambda}^{[p]}) \in \mathscr{H}_{d,\mathbf{b}}$ such that $S_{\mathbf{b}}(\boldsymbol{\lambda}) \cong s_{\mathbf{b}}(\boldsymbol{\lambda})\mathscr{H}_{d,\mathbf{b}}$. Explicitly, $s(\boldsymbol{\lambda}^{[i]}) = u^+_{\boldsymbol{\mu}^{[i]}}(\varepsilon^i\mathbf{Q}')y_{\boldsymbol{\mu}^{[i]}}T_{w(\boldsymbol{\mu})}x_{\boldsymbol{\lambda}^{[i]}}u^+_{\boldsymbol{\lambda}^{[i]}}$ where $\boldsymbol{\mu}^{[i]}$ is the multipartition conjugate to $\boldsymbol{\lambda}^{[i]}$, for $1 \le i \le p$. By Lemma 3.3, we have that

$$S(\boldsymbol{\lambda}) \cong \mathtt{H}_{\mathbf{b}}(S_{\mathbf{b}}(\boldsymbol{\lambda})) \cong s_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot v_{\mathbf{b}}\mathscr{H}_{r,n}. \tag{3.22}$$

Henceforth, we identify $S(\boldsymbol{\lambda})$ with $s_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot V_{\mathbf{b}}$ and $S_{\mathbf{b}}(\boldsymbol{\lambda})$ with $s_{\mathbf{b}}(\boldsymbol{\lambda})\mathscr{H}_{d,\mathbf{b}}$ via these isomorphisms. Observe that $\mathtt{H}_{\mathbf{b}}(S_{\mathbf{b}}(\boldsymbol{\lambda})) = S(\boldsymbol{\lambda})$ with these identifications.

Definition 3.23. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Define*

$$M_{\mathbf{b}}(\boldsymbol{\lambda}) = u^+_{\boldsymbol{\lambda},\mathbf{b}}x_{\boldsymbol{\lambda},\mathbf{b}}\mathscr{H}_{d,\mathbf{b}} \qquad \textit{and} \qquad M^{\boldsymbol{\lambda}}_{\mathbf{b}} = \mathtt{H}_{\mathbf{b}}\big(M_{\mathbf{b}}(\boldsymbol{\lambda})\big).$$

The definitions above apply equally well to $\mathscr{H}_{r,n}$-modules by taking $p = 1$. In particular, we have elements $u^+_{\boldsymbol{\lambda}}$ and $x_{\boldsymbol{\lambda}}$ in $\mathscr{H}_{r,n}$ and an $\mathscr{H}_{r,n}$-module $M(\boldsymbol{\lambda}) = u^+_{\boldsymbol{\lambda}}x_{\boldsymbol{\lambda}}\mathscr{H}_{r,n}$. Using the definitions it is easy to check that $x_{\boldsymbol{\lambda}} = \Theta_{\mathbf{b}}(x_{\boldsymbol{\lambda},\mathbf{b}})$ and that $u^+_{\boldsymbol{\lambda}} = u^+_{\mathbf{b}}\Theta_{\mathbf{b}}(u^+_{\boldsymbol{\lambda},\mathbf{b}})$, where $u^+_{\mathbf{b}}$ is the element introduced in (2.27). It follows that $M^{\boldsymbol{\lambda}}_{\mathbf{b}} = v^+_{\mathbf{b}}M(\boldsymbol{\lambda})$. Hence, in general, $M^{\boldsymbol{\lambda}}_{\mathbf{b}}$ is a proper submodule of $V_{\mathbf{b}}$.

We can now prove the promised result about σ-twisted modules.

Proposition 3.24. *Let $\mathbf{b} \in \mathscr{C}_{p,n}$ and $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Suppose that $\mathbf{Q}$ is (ε, q)-separated over K. Then*

$$\big(M^{\boldsymbol{\lambda}}_{\mathbf{b}}\big)^\sigma \cong M^{\boldsymbol{\lambda}\langle -1\rangle}_{\mathbf{b}\langle -1\rangle} \quad \textit{and} \quad S(\boldsymbol{\lambda})^\sigma \cong S(\boldsymbol{\lambda}\langle -1\rangle).$$

Moreover, if $\boldsymbol{\lambda} \in \mathscr{K}_{r,n}$ *then* $D(\boldsymbol{\lambda})^\sigma \cong D(\boldsymbol{\lambda}\langle -1\rangle)$.

Proof. We have that $\sigma\big(u^+_{\boldsymbol{\lambda}^{[t]}}(\varepsilon^t\mathbf{Q})\big) = \varepsilon^{k_t} u^+_{\boldsymbol{\lambda}^{[t]}}(\varepsilon^{t-1}\mathbf{Q})$, for some integer k_t, exactly as in Lemma 3.18. From the definitions, $F^\sigma_{\mathbf{b}}\big(M_{\mathbf{b}}(\boldsymbol{\lambda})\big) \cong M_{\mathbf{b}\langle -1\rangle}(\boldsymbol{\lambda}\langle -1\rangle)$. Therefore, using Proposition 3.21,

$$\begin{aligned}\big(M^{\boldsymbol{\lambda}}_{\mathbf{b}}\big)^\sigma &= \mathtt{F}^\sigma\big(\mathtt{H}_{\mathbf{b}}\big(M_{\mathbf{b}}(\boldsymbol{\lambda})\big)\big) \cong \mathtt{H}_{\mathbf{b}\langle -1\rangle}\big(F^\sigma_{\mathbf{b}}\big(M_{\mathbf{b}}(\boldsymbol{\lambda})\big)\big)\\ &\cong \mathtt{H}_{\mathbf{b}\langle -1\rangle}\big(M_{\mathbf{b}\langle -1\rangle}(\boldsymbol{\lambda}\langle -1\rangle)\big) \cong M^{\boldsymbol{\lambda}\langle -1\rangle}_{\mathbf{b}\langle -1\rangle},\end{aligned}$$

giving the first isomorphism. A similar argument shows that $S(\boldsymbol{\lambda})^\sigma \cong S(\boldsymbol{\lambda}\langle -1\rangle)$.

Finally, if $\boldsymbol{\lambda}$ is Kleshchev then $D(\boldsymbol{\lambda}) \neq 0$ and there is a short exact sequence

$$0 \longrightarrow \operatorname{rad} S(\boldsymbol{\lambda}) \longrightarrow S(\boldsymbol{\lambda}) \longrightarrow D(\boldsymbol{\lambda}) \longrightarrow 0.$$

The functor $\mathtt{F}^\sigma$ is exact, and $D(\boldsymbol{\lambda}\langle -1\rangle)$ is the head of $S(\boldsymbol{\lambda}\langle -1\rangle)$, so $D(\boldsymbol{\lambda})^\sigma \cong D(\boldsymbol{\lambda}\langle -1\rangle)$ because $S(\boldsymbol{\lambda})^\sigma \cong S(\boldsymbol{\lambda}\langle -1\rangle)$ by the last paragraph. (Note that $\boldsymbol{\lambda}$ is Kleshchev if and only if $\boldsymbol{\lambda}\langle -1\rangle$ is Kleshchev by Lemma 3.3(c).) □

As σ is trivial on $\mathscr{H}_{r,p,n}$, Lemma 3.19 and Proposition 3.24 imply the following.

Corollary 3.25. *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated over* K *and that* $\mathbf{b} \in \mathscr{C}_{p,n}$, $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ *and* $t \in \mathbb{Z}$. *Then:*

a) $V_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong V_{\mathbf{b}\langle t\rangle}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$,
b) $M^{\boldsymbol{\lambda}}_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong M^{\boldsymbol{\lambda}\langle t\rangle}_{\mathbf{b}\langle t\rangle}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$,
c) $S(\boldsymbol{\lambda})\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong S(\boldsymbol{\lambda}\langle t\rangle)\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$, *and,*
d) *if* $\boldsymbol{\lambda} \in \mathscr{K}_{r,n}$ *then* $D(\boldsymbol{\lambda})\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong D(\boldsymbol{\lambda}\langle t\rangle)\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$.

3.5. **Shifting homomorphisms.** In this section we show that our ordering of the cyclotomic parameters $\mathbf{Q}^{\vee\varepsilon}$ in (2.1) implies the existence of some isomorphisms between Specht modules. The shifting homomorphisms are, ultimately, what allow us to construct the irreducible $\mathscr{H}_{r,p,n}$ modules — and hence prove Theorem C. These result also underpin our calculation of the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ and, consequently, our proof of Theorem D.

Extending the notation that we used for the modules $V^{(t)}_{\mathbf{b}}$, for each multipartition $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ let $S(\boldsymbol{\lambda})^{(t)}$ be the Specht module for $\mathscr{H}_{r,n}$ which is defined with respect to the ordered parameters $\varepsilon^t\mathbf{Q}^{\vee\varepsilon}$ (rather than $\mathbf{Q}^{\vee\varepsilon}$). Then $S(\boldsymbol{\lambda}) \cong S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$ as $\mathscr{H}_{r,n}$-modules and $S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$ is a submodule of $V^{(t)}_{\mathbf{b}\langle t\rangle}$. The following result makes this more explicit.

Lemma 3.26. *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated over* K *and that* $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, *for* $\mathbf{b} \in \mathscr{C}_{p,n}$, *and* $1 \le t \le p$. *Then*

$$Y_t \dots Y_1 S(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$$

as subsets of $\mathscr{H}_{r,n}$.

Proof. As we have already observed, left multiplication by $Y_p \dots Y_1$ is invertible by Lemma 2.28 and Lemma 2.25. Therefore, $Y_t \dots Y_1 S(\boldsymbol{\lambda}) \cong S(\boldsymbol{\lambda})$ as a right $\mathscr{H}_{r,n}$-modules, so it is enough to show that $Y_t \dots Y_1 S(\boldsymbol{\lambda}) \subseteq S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$. Recall from before Definition 3.23 that we are identifying $S_{\mathbf{b}}(\boldsymbol{\lambda})$ with the ideal $S_{\mathbf{b}}(\boldsymbol{\lambda}) = s_{\mathbf{b}}(\boldsymbol{\lambda})\mathscr{H}_{d,\mathbf{b}}$

and $S(\boldsymbol{\lambda}) = s_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot V_{\mathbf{b}}$. Using Lemma 2.20 we compute

$$\begin{aligned} Y_t \dots Y_1\big(s_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot v_{\mathbf{b}}\big) &= Y_t \dots Y_1 v_{\mathbf{b}} \Theta_{\mathbf{b}}\big(s_{\mathbf{b}}(\boldsymbol{\lambda})\big) \\ &= \widehat{\Theta}_{\mathbf{b}\langle t\rangle}\big(s_{\mathbf{b}\langle t\rangle}(\boldsymbol{\lambda}\langle t\rangle)\big) Y_t \dots Y_1 v_{\mathbf{b}} \\ &= s_{\mathbf{b}\langle t\rangle}(\boldsymbol{\lambda}\langle t\rangle) \cdot v^{(t)}_{\mathbf{b}\langle t\rangle} Y_t^* \dots Y_1^*, \end{aligned}$$

the last equality following from Corollary 2.15. Hence, $Y_t \dots Y_1 S(\boldsymbol{\lambda}) \subseteq S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$ as we needed to show. □

Fix $\mathbf{b} \in \mathscr{C}_{p,n}$ and $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ and suppose that $\boldsymbol{\lambda} = \boldsymbol{\lambda}\langle m\rangle$, for some integer $1 \le m \le p$ with m dividing p. Then $\mathbf{b} = \mathbf{b}\langle m\rangle$ and σ^m is an automorphism of $\mathscr{H}_{r,n}$ of order $\frac{p}{m}$. Set

$$\check{\mathbf{Q}} = \big(Q_1, Q_1\varepsilon, \cdots, Q_1\varepsilon^{m-1}, Q_2, \cdots, Q_2\varepsilon^{m-1}, \cdots, Q_d, \cdots, Q_d\varepsilon^{m-1}\big).$$

Then $\mathscr{H}_{r,n} = \mathscr{H}_{r,n}(\mathbf{Q}^{\vee\varepsilon}) = \mathscr{H}_{r,n}(\check{\mathbf{Q}}^{\vee\varepsilon^m})$. By definition, $\mathscr{H}_{r,\frac{p}{m},n} = \mathscr{H}_{r,\frac{p}{m},n}(\check{\mathbf{Q}})$ is the subalgebra of $\mathscr{H}_{r,n}$ generated by $T_0^{p/m}, T_1, \cdots, T_{n-1}$, so that

$$\mathscr{H}_{r,\frac{p}{m},n} \cong \{\, h \in \mathscr{H}_{r,n} \mid h = \sigma^m(h) \,\}. \tag{3.27}$$

This observation will be useful below.

For $0 \le t < \frac{p}{m}$ we now consider the modules $V_{\mathbf{b}}^{(tm)}$ and $S(\boldsymbol{\lambda})^{(tm)}$. Then, by definition, $S(\boldsymbol{\lambda})^{(tm)}$ is a submodule of $V_{\mathbf{b}}^{(tm)}$, Further, by Lemma 3.18 and Proposition 3.24,

$$\big(V_{\mathbf{b}}^{(tm+m)}\big)^{\sigma^{-m}} = V_{\mathbf{b}}^{(tm)} \quad \text{and} \quad \big(S(\boldsymbol{\lambda})^{(tm+m)}\big)^{\sigma^{-m}} = S(\boldsymbol{\lambda})^{(tm)}.$$

Motivated by Definition 2.16, define

$$Y_{t,m} = Y_{tm+m} \dots Y_{tm+2} Y_{tm+1},$$

for $0 \le t < \frac{p}{m}$, and let $\theta'_{t,m} \colon V_{\mathbf{b}}^{(tm)} \longrightarrow V_{\mathbf{b}}^{(tm+m)}$ be the map $\theta'_{t,m}(v) = Y_{t,m} v$, for $v \in V_{\mathbf{b}}^{(tm)}$.

Definition 3.28 ((Shifting homomorphisms)). *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and that $\mathbf{b} = \mathbf{b}\langle m\rangle$ for some $1 \le m \le p$ with m dividing p. For $0 \le t < \frac{p}{m}$ define $\theta_{t,m} = \sigma^m \circ \theta'_{t,m}$.*

Lemma 3.29. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$, with $\mathbf{b} = \mathbf{b}\langle m\rangle$ for some $1 \le m \le p$ with m dividing p, and suppose that $0 \le t < \frac{p}{m}$. Then $\theta_{t,m} \in \operatorname{End}_{\mathscr{H}_{r,p/m,n}}\big(V_{\mathbf{b}}^{(tm)}\big)$.*

Proof. By Definition 2.16 and the remarks above, $\theta_{t,m} \in \operatorname{End}_R\big(V_{\mathbf{b}}^{(tm)}\big)$ since $\mathbf{b} = \mathbf{b}\langle m\rangle$. Moreover, if $v \in V_{\mathbf{b}}^{(tm)}$ and $h \in \mathscr{H}_{r,n}$ then

$$\theta_{t,m}(vh) = \sigma^m\big(\theta'_{t,m}(vh)\big) = \sigma^m\big(\theta'_{t,m}(v)\big)\sigma^m(h),$$

since $\theta'_{t,m}$ is an $\mathscr{H}_{r,n}$-module homomorphism by Definition 2.16. Therefore, $\theta_{t,m}(vh)$ is an $\mathscr{H}_{r,p/m,n}$-module homomorphism since $\mathscr{H}_{r,\frac{p}{m},n} = \mathscr{H}_{r,n}^{\sigma^m}$ by (3.27). □

3.6. **Seminormal forms and roots of $\mathfrak{f}_{\boldsymbol{\lambda}}$.** In this section we show that if $\boldsymbol{\lambda} = \boldsymbol{\lambda}\langle m\rangle$, for an integer m dividing p such that $1 \le m \le p$, then there exists a scalar $\mathfrak{f}^{(1)}_{\boldsymbol{\lambda}}$ such that $\mathfrak{f}_{\boldsymbol{\lambda}} = \varepsilon^{mnd(l-1)/2}\big(\mathfrak{f}^{(1)}_{\boldsymbol{\lambda}}\big)^l$, where $l = p/m$ as in Theorem B. By relating $\mathfrak{f}_{\boldsymbol{\lambda}}$ to the shifting homomorphisms we will show that this factorization of $\mathfrak{f}_{\boldsymbol{\lambda}}$ corresponds to a factorization of the endomorphism of $S(\boldsymbol{\lambda})$ given by left multiplication by $z_{\mathbf{b}}$.

Recall that $\mathcal{A} = \mathbb{Z}[\dot\varepsilon, \dot q^{\pm1}, \dot Q_1^{\pm1}, \dots, \dot Q_d^{\pm1}, A(\dot\varepsilon, \dot q, \dot{\mathbf{Q}})^{-1}]$ and that $\mathcal{F}$ is the field of fractions of $\mathcal{A}$. As saw in Section 3.2, the algebra $\mathscr{H}^{\mathcal{F}}_{r,n}$ is semisimple. Note that $\dot{\mathbf{Q}}$ is $(\dot\varepsilon, \dot q)$-separated over $\mathcal{F}$ so we can apply all of our previous results.

Fix $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ and an integer m such that $\boldsymbol{\lambda} = \boldsymbol{\lambda}\langle m\rangle$ and $1 \le m \le p$ and $m \mid p$. Let $l = p/m$. Since $\mathscr{H}^{\mathcal{F}}_{r,n}$ is semisimple the Specht module $S(\boldsymbol{\lambda}) = S^{\mathcal{F}}(\boldsymbol{\lambda})$ is irreducible and has, as we now recall, a seminormal representation over $\mathcal{F}$. First we need some notation.

Recall from Section 3.1 that $\operatorname{Std}(\boldsymbol{\lambda})$ is the set of standard $\boldsymbol{\lambda}$-tableaux. Each standard tableau $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ is an r-tuple $\mathfrak{s} = (\mathfrak{s}^{(1)}, \dots, \mathfrak{s}^{(r)})$ of standard tableaux. Extending the notation for $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{[1]}, \dots, \boldsymbol{\lambda}^{[p]})$ write $\mathfrak{s} = (\mathfrak{s}^{[1]}, \dots, \mathfrak{s}^{[p]})$, where $\mathfrak{s}^{[j]} = (\mathfrak{s}^{[jd-d+1]}, \dots, \mathfrak{s}^{[jd]})$ is a $\boldsymbol{\lambda}^{[j]}$-tableau for $1 \le j \le p$. Similarly, if $z \in \mathbb{Z}$ define $\mathfrak{s}\langle z\rangle = (\mathfrak{s}^{[z+1]}, \dots, \mathfrak{s}^{[z+p]})$ where, as usual, we set $\mathfrak{s}^{[j+kp]} = \mathfrak{s}^{[j]}$ for $1 \le j \le p$ and $k \in \mathbb{Z}$.

If $1 \le k \le n$ and $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ define the **content** of k in $\mathfrak{t}$ to be

$$\operatorname{cont}_{\mathfrak{s}}(k) = \dot\varepsilon^{j}\dot q^{b-a}\dot Q_c \in \mathcal{F},$$

if k appears in row a and column b of $\mathfrak{s}^{(c+jd)}$. The following useful fact is easily proved by induction on n.

Lemma 3.30 ((cf. [26, Lemma 3.12]))**.** *Suppose that $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu})$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{r,n}$. Then $\mathfrak{s} = \mathfrak{t}$ if and only if $\operatorname{cont}_{\mathfrak{s}}(k) = \operatorname{cont}_{\mathfrak{t}}(k)$, for $1 \le k \le n$.*

If $\mathfrak{s}$ is a standard $\boldsymbol{\lambda}$-tableau and $1 \le i < n$ let $\mathfrak{s}(i, i+1)$ be the tableau obtained by interchanging the positions of i and $i+1$ in $\mathfrak{s}$. Then $\mathfrak{s}(i, i+1)$ is a standard $\boldsymbol{\lambda}$-tableau unless i and $i+1$ are either in the same row or in the same column.

Lemma 3.31 ((Ariki-Koike [4, Theorem 3.7]))**.** *Let $V(\boldsymbol{\lambda})$ be the $\mathcal{F}$-vector space with basis $\{\, v_{\mathfrak{s}} \mid \mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$. Then $V(\boldsymbol{\lambda})$ becomes an $\mathscr{H}^{\mathcal{F}}_{r,n}$-module with $\mathscr{H}^{\mathcal{F}}_{r,n}$-action, for $1 \le k \le n$ and $1 \le i < n$, given by*

$$v_{\mathfrak{s}} L_k = \operatorname{cont}_{\mathfrak{s}}(k) v_{\mathfrak{s}} \quad \text{and} \quad v_{\mathfrak{s}} T_i = \beta_{\mathfrak{s}}(i) v_{\mathfrak{s}} + \big(1 + \beta_{\mathfrak{s}}(i)\big) v_{\mathfrak{t}},$$

where $\mathfrak{t} = \mathfrak{s}(i, i+1)$, $v_{\mathfrak{t}} = 0$ if $\mathfrak{t}$ is not standard and

$$\beta_{\mathfrak{s}}(i) = \frac{(\dot q - 1)\operatorname{cont}_{\mathfrak{t}}(i)}{(\operatorname{cont}_{\mathfrak{t}}(i) - \operatorname{cont}_{\mathfrak{s}}(i))}.$$

Moreover, $V(\boldsymbol{\lambda}) \cong S^{\mathcal{F}}(\boldsymbol{\lambda})$ as $\mathscr{H}^{\mathcal{F}}_{r,n}$-modules.

The module $V(\boldsymbol{\lambda})$ is a **seminormal form** for $S^{\mathcal{F}}(\boldsymbol{\lambda})$.

Recall that we have fixed integers m and $l = p/m$ such that $m \mid p$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}\langle m\rangle$. Thus, $S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)} \cong S^{\mathcal{F}}(\boldsymbol{\lambda})$, for $0 \le t < l = p/m$. By (3.22),

$$S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)} = s_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot v^{(tm)}_{\mathbf{b}} \mathscr{H}^{\mathcal{F}}_{r,n}.$$

For convenience, we set $v^{(tm)}_{\mathfrak{t}^{\boldsymbol{\lambda}}} = s_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot v^{(tm)}_{\mathbf{b}} \in \mathscr{H}^{\mathcal{F}}_{r,n}$.

Recall from Section 3.2 that $\mathfrak{t}^{\boldsymbol{\lambda}}$ is the standard $\boldsymbol{\lambda}$-tableau which has the numbers $1, 2, \dots, n$ entered in order from left to right along the rows of its first component, then its second component and so on.

Lemma 3.32. *Suppose that $0 \le t < l$. Then*

$$v^{(tm)}_{\mathfrak{t}^{\boldsymbol{\lambda}}} L_k = \mathrm{cont}_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}(k) v^{(tm)}_{\mathfrak{t}^{\boldsymbol{\lambda}}},$$

for $1 \le k \le n$.

Proof. It suffices to consider the case $t = 0$ when the result is effectively a restatement of [28, Proposition 3.13]. Alternatively, this can be proved using Du and Rui's proof [14, Theorem 2.9] that the Specht module $S(\boldsymbol{\lambda})$ is isomorphic to the corresponding cell module from [11] together with the description of the action of $L_1, \dots, L_n$ on the standard basis of the cell modules from [26, Proposition 3.7]. □

Corollary 3.33. *Suppose that $0 \le t < l$. Then there exists a unique $\mathscr{H}^{\mathcal{F}}_{r,n}$-module isomorphism*

$$\varphi^{(tm)}_{\boldsymbol{\lambda}}: V(\boldsymbol{\lambda}) \xrightarrow{\simeq} S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$$

such that $\varphi^{(tm)}_{\boldsymbol{\lambda}}(v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}) = v^{(tm)}_{\mathfrak{t}^{\boldsymbol{\lambda}}}$.

Proof. By the Lemma, $v^{(tm)}_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ is a simultaneous eigenvector for $L_1, \dots, L_n$ with the eigenvalues being given by the content functions $\mathrm{cont}_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle tm\rangle}(k)$, for $1 \le k \le n$. By Proposition 3.31 the corresponding simultaneous eigenspace in $V(\boldsymbol{\lambda})$ is $\mathcal{F} v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}$, so any $\mathscr{H}^{\mathcal{F}}_{r,n}$-module isomorphism from $V(\boldsymbol{\lambda})$ to $S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$ must send $v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}$ to a scalar multiple of $v^{(tm)}_{\mathfrak{t}^{\boldsymbol{\lambda}}}$. As $V(\boldsymbol{\lambda}) \cong S^{\mathcal{F}}(\boldsymbol{\lambda}) \cong S^{\mathcal{F}}(\boldsymbol{\lambda}\langle tm\rangle)^{(tm)} = S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$, by renormalising any isomorphism $V(\boldsymbol{\lambda}) \longrightarrow S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$ we get the result. □

Suppose that $0 \le t < l$. For each standard $\boldsymbol{\lambda}$-tableau $\mathfrak{s}$ set $v^{(tm)}_{\mathfrak{s}} = \varphi^{(tm)}_{\boldsymbol{\lambda}}(v_{\mathfrak{s}\langle -tm\rangle})$. Then $\{\, v^{(tm)}_{\mathfrak{s}} \mid \mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})\,\}$ is a Young seminormal basis of $S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$ and, by construction,

$$v^{(tm)}_{\mathfrak{s}} L_k = \varphi^{(tm)}_{\boldsymbol{\lambda}}(v_{\mathfrak{s}\langle tm\rangle}) L_k = \mathrm{cont}_{\mathfrak{s}\langle tm\rangle}(k) v^{(tm)}_{\mathfrak{s}},$$

for $1 \le k \le n$. Recall from Lemma 3.26 that $Y_{t,m} S(\boldsymbol{\lambda})^{(tm)} = S(\boldsymbol{\lambda})^{(tm+m)}$. We can now describe the map given by left multiplication by $Y_{t,m}$ more explicitly.

Proposition 3.34. *Suppose that $0 \le t \le m$ and $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then there exists a scalar $\dot{\mathfrak{f}}^{(t+1:m)}_{\boldsymbol{\lambda}}(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}}) \in \mathcal{F}$ such that*

$$Y_{t,m} v^{(tm)}_{\mathfrak{s}\langle m\rangle} = \dot{\mathfrak{f}}^{(t+1:m)}_{\boldsymbol{\lambda}} v^{(tm+m)}_{\mathfrak{s}},$$

for all $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})$.

Proof. By definition, if $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $v^{(tm+m)}_{\mathfrak{s}} L_k = \mathrm{cont}_{\mathfrak{s}\langle tm+m\rangle}(k) v^{(tm+m)}_{\mathfrak{s}}$, for $1 \le k \le n$. The same statement holds true for $Y_{t,m} v^{(tm)}_{\mathfrak{s}\langle m\rangle}$, so by construction $Y_{t,m} v^{(tm)}_{\mathfrak{s}\langle m\rangle}$ must be a scalar multiple of $v^{(tm+m)}_{\mathfrak{s}}$. By direct calculation the map which sends $v^{(tm)}_{\mathfrak{s}\langle m\rangle}$ to $v^{(tm+m)}_{\mathfrak{s}}$, for each $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})$, defines an $\mathscr{H}^{\mathcal{F}}_{r,n}$-isomorphism. By Schur's Lemma this scalar is independent of $\mathfrak{s}$ so the Lemma follows. □

We write $\dot{\mathfrak{f}}^{(t)}_{\boldsymbol{\lambda}} = \dot{\mathfrak{f}}^{(t:m)}_{\boldsymbol{\lambda}}(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}})$ if m is clear from context. It is tempting to say that $\dot{\mathfrak{f}}^{(t)}_{\boldsymbol{\lambda}} \in \mathcal{A}$ since left multiplication by $Y_{t,m}$ is defined over $\mathcal{A}$, however, the construction of the basis $\{v^{(tm)}_{\mathfrak{s}}\}$ is only valid over $\mathcal{F}$. Nonetheless, we will show below that $\dot{\mathfrak{f}}^{(t)}_{\boldsymbol{\lambda}} \in \mathcal{A}$ using the fact that $\mathcal{A}$ is integrally closed in $\mathcal{F}$, .

Lemma 3.35. *Let $\varphi\colon V(\boldsymbol\lambda)\longrightarrow V(\boldsymbol\lambda)$ be the $\mathcal F$-linear map such that*

$$\varphi(v_{\mathfrak s}) = v_{\mathfrak s\langle m\rangle}, \qquad \textit{for all } \mathfrak s\in \operatorname{Std}(\boldsymbol\lambda).$$

Then φ is an $\mathscr H_{\dot q}^{\mathcal F}(\mathfrak S_n)$-module homomorphism. Moreover, $\varphi(vx)=\varphi(v)\sigma^m(x)$, for all $v\in V(\boldsymbol\lambda)$ and $x\in\mathscr H_{r,n}^{\mathcal F}$. Hence, φ is an $\mathscr H_{r,p/m,n}^{\mathcal F}$-module homomorphism.

Proof. Suppose that $\mathfrak s\in\operatorname{Std}(\boldsymbol\lambda)$ and $1\le i<n$ and let $\mathfrak t=\mathfrak s(i,i+1)$. Then, by Lemma 3.31,

$$\begin{aligned}\varphi(v_{\mathfrak s}T_i) &= \beta_{\mathfrak s}(i)\varphi(v_{\mathfrak s}) + (1+\beta_{\mathfrak s}(i))\varphi(v_{\mathfrak t}) = \beta_{\mathfrak s}(i)v_{\mathfrak s\langle m\rangle} + (1+\beta_{\mathfrak s}(i))v_{\mathfrak t\langle m\rangle}\\ &= v_{\mathfrak s\langle m\rangle}T_i = \varphi(v_{\mathfrak s})T_i,\end{aligned}$$

where the second last equality follows because $\beta_{\mathfrak s}(i)=\beta_{\mathfrak s\langle m\rangle}(i)$. Hence, φ is a $\mathscr H_q(\mathfrak S_n)$-homomorphism. To prove the second claim it is enough to show that $\varphi(v_{\mathfrak s}L_k)=\dot\varepsilon^m v_{\mathfrak s\langle m\rangle}L_k$, for all $\mathfrak s\in\operatorname{Std}(\boldsymbol\lambda)$ and $1\le k\le n$. This is immediate because $\operatorname{cont}_{\mathfrak s\langle m\rangle}(k)=\dot\varepsilon^{-m}\operatorname{cont}_{\mathfrak s}(k)$ by Lemma 3.32. □

Corollary 3.36. *Suppose that $0\le t<l$ and that $\mathfrak s\in\operatorname{Std}(\boldsymbol\lambda)$. Then*

$$\sigma^m\big(v_{\mathfrak s}^{(tm)}\big)=\dot\varepsilon^{-dmn}v_{\mathfrak s}^{(tm-m)}.$$

Proof. First note that $\sigma^m(v_{\mathfrak t^{\boldsymbol\lambda}}^{(tm)})=\dot\varepsilon^{-dmn}v_{\mathfrak t^{\boldsymbol\lambda}}^{(tm-m)}$ because

$$\sigma^m(v_{\mathfrak t^{\boldsymbol\lambda}}^{(tm)})=\sigma^m\big(s_{\mathbf b}(\boldsymbol\lambda)\cdot v_{\mathbf b}^{(tm)}\big)=\dot\varepsilon^{-dmn}s_{\mathbf b}(\boldsymbol\lambda)\cdot v_{\mathbf b}^{(tm-m)}=\dot\varepsilon^{-dmn}v_{\mathfrak t^{\boldsymbol\lambda}}^{(tm-m)},$$

by Lemma 3.18. Therefore, writing $v_{\mathfrak s}^{(tm)}=v_{\mathfrak t^{\boldsymbol\lambda}}^{(tm)}h=\varphi_{\boldsymbol\lambda}^{(tm)}(v_{\mathfrak t^{\boldsymbol\lambda}\langle -tm\rangle}h)$ we have $v_{\mathfrak t^{\boldsymbol\lambda}\langle -tm\rangle}h=v_{\mathfrak s\langle -tm\rangle}$, and so

$$\begin{aligned}\sigma^m\big(v_{\mathfrak s}^{(tm)}\big) &= \sigma^m(v_{\mathfrak t^{\boldsymbol\lambda}}^{(tm)})\sigma^m(h)=\dot\varepsilon^{-dmn}v_{\mathfrak t^{\boldsymbol\lambda}}^{(tm-m)}\sigma^m(h)\\ &= \dot\varepsilon^{-dmn}\varphi_{\boldsymbol\lambda}^{(tm-m)}\big(v_{\mathfrak t^{\boldsymbol\lambda}\langle m-tm\rangle}\sigma^m(h)\big)=\dot\varepsilon^{-dmn}\varphi_{\boldsymbol\lambda}^{(tm-m)}\big(\varphi(v_{\mathfrak t^{\boldsymbol\lambda}\langle -tm\rangle}h)\big)\\ &= \dot\varepsilon^{-dmn}\varphi_{\boldsymbol\lambda}^{(tm-m)}\big(\varphi(v_{\mathfrak s\langle -tm\rangle})\big)=\dot\varepsilon^{-dmn}\varphi_{\boldsymbol\lambda}^{(tm-m)}\big(v_{\mathfrak s\langle m-tm\rangle}\big)\\ &= \dot\varepsilon^{-dmn}v_{\mathfrak s}^{(tm-m)},\end{aligned}$$

as required. □

Theorem 3.37. *Suppose that $\boldsymbol\lambda\in\mathscr P_{d,\mathbf b}$ be a multipartition such that $\boldsymbol\lambda=\boldsymbol\lambda\langle m\rangle$, for some $\mathbf b\in\mathscr C_{p,n}$ and $1\le m\le p$ with $m\mid p$. Set $l=p/m$. Then*

$$\dot{\mathfrak f}_{\boldsymbol\lambda}=\dot{\mathfrak f}_{\boldsymbol\lambda}^{(1)}\dots\dot{\mathfrak f}_{\boldsymbol\lambda}^{(l)}=\dot\varepsilon^{\frac12 dmn(1-l)}\big(\dot{\mathfrak f}_{\boldsymbol\lambda}^{(1)}\big)^l.$$

Consequently, $\dot{\mathfrak f}_{\boldsymbol\lambda}^{(t)}\in\mathcal A$ for $1\le t\le l$.

Proof. By Lemma 3.4 and Proposition 3.34, if $\mathfrak s\in\operatorname{Std}(\boldsymbol\lambda)$ then

$$\begin{aligned}\dot{\mathfrak f}_{\boldsymbol\lambda}v_{\mathfrak s}^{(0)} &= Y_p\dots Y_1v_{\mathfrak s}^{(0)}=Y_{l-1,m}\dots Y_{0,m}v_{\mathfrak s}^{(0)}\\ &= \dot{\mathfrak f}_{\boldsymbol\lambda}^{(1)}Y_{l-1,m}\dots Y_{1,m}v_{\mathfrak s\langle -m\rangle}^{(m)}=\dots=\dot{\mathfrak f}_{\boldsymbol\lambda}^{(1)}\dots\dot{\mathfrak f}_{\boldsymbol\lambda}^{(l)}v_{\mathfrak s}^{(p)}.\end{aligned}$$

Therefore, $\dot{\mathfrak f}_{\boldsymbol\lambda}=\dot{\mathfrak f}_{\boldsymbol\lambda}^{(1)}\dots\dot{\mathfrak f}_{\boldsymbol\lambda}^{(l)}$, since $v_{\mathfrak s}^{(p)}=v_{\mathfrak s}^{(0)}$. This proves the first claim.

For the second claim, observe that by Lemma 3.18

$$\sigma^m(Y_{t,m})=\dot\varepsilon^{(p-1)dm\mathbf b_1^m}Y_{t-1,m}=\dot\varepsilon^{-dmn/l}Y_{t-1,m},$$

since $\dot\varepsilon^p=1$ and $l\mathbf{b}_1^m=\mathbf{b}_1^p=n$. Therefore,

$$\begin{aligned}\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t+1)}v_{\mathfrak{t}^{\boldsymbol\lambda}}^{(tm+m)}&=Y_{t,m}v_{\mathfrak{t}^{\boldsymbol\lambda}\langle m\rangle}^{(tm)}=\sigma^{-m}\big(\sigma^m(Y_{t,m}v_{\mathfrak{t}^{\boldsymbol\lambda}\langle m\rangle}^{(tm)})\big)\\&=\dot\varepsilon^{-dmn(1+1/l)}\sigma^{-m}\big(Y_{t-1,m}v_{\mathfrak{t}^{\boldsymbol\lambda}\langle m\rangle}^{(tm-m)}\big)\\&=\dot\varepsilon^{-dmn(1+1/l)}\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t)}\sigma^{-m}\big(v_{\mathfrak{t}^{\boldsymbol\lambda}}^{(tm)}\big)\\&=\dot\varepsilon^{-dmn/l}\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t)}v_{\mathfrak{t}^{\boldsymbol\lambda}}^{(tm+m)}\end{aligned}$$

Therefore, $\mathfrak{f}_{\boldsymbol\lambda}^{(t+1)}=\dot\varepsilon^{-dmn/l}\mathfrak{f}_{\boldsymbol\lambda}^{(t)}=\dots=\dot\varepsilon^{-tdmn/l}\mathfrak{f}_{\boldsymbol\lambda}^{(1)}$. The second claim follows.

To complete the proof observe, for example using [15, page 138, Exercise 4.18 and 4.21], that the ring $\mathcal{A}$ is an integrally closed domain. Therefore, $\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(1)}\in\mathcal{A}$ because $\dot{\mathfrak{f}}_{\boldsymbol\lambda}$ and $\big(\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(1)}\big)^l=\dot\varepsilon^{\frac12 dmn(l-1)}\dot{\mathfrak{f}}_{\boldsymbol\lambda}$ both belong to $\mathcal{A}$ by Proposition 3.4. Hence, $\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t)}\in\mathcal{A}$, for $1\le t\le m$, completing the proof. □

Henceforth, let $\mathfrak{f}_{\boldsymbol\lambda}^{(t)}$ be the value of $\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t)}$ at $(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})=(\varepsilon,q,\mathbf{Q})$, for each integer $1\le t\le l$.

Corollary 3.38. *Suppose that $\mathbf{Q}$ is (ε,q)-separated over R and let $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf{b}}$ be a multipartition such that $\boldsymbol\lambda=\boldsymbol\lambda\langle m\rangle$, for some $\mathbf{b}\in\mathscr{C}_{p,n}$ and $1\le m\le p$ with $m\mid p$. Set $l=p/m$. Then $\mathfrak{f}_{\boldsymbol\lambda}=\mathfrak{f}_{\boldsymbol\lambda}^{(1)}\dots\mathfrak{f}_{\boldsymbol\lambda}^{(l)}=\varepsilon^{\frac12 dmn(1-l)}\big(\mathfrak{f}_{\boldsymbol\lambda}^{(1)}\big)^l$.*

Combining Corollary 3.38 with Proposition 3.4 and Theorem 3.7 we have proved Theorem B from the introduction.

Recall from the introduction that $\mathsf{o}_{\boldsymbol\lambda}=\min\{\,k\ge1\mid \boldsymbol\lambda^{[k+t]}=\boldsymbol\lambda^{[t]},\text{ for all }t\in\mathbb{Z}\,\}$, and that $p_{\boldsymbol\lambda}=p/\mathsf{o}_{\boldsymbol\lambda}$. Note that $\mathsf{o}_{\boldsymbol\lambda}$ divides p so that $p_{\boldsymbol\lambda}$ is an integer.

Definition 3.39. *Suppose that $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$. Define $\dot{\mathfrak{g}}_{\boldsymbol\lambda}=\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(1:\mathsf{o}_{\boldsymbol\lambda})}$. If $\mathbf{Q}$ is (ε,q)-separated let $\mathfrak{g}_{\boldsymbol\lambda}=\dot{\mathfrak{g}}_{\boldsymbol\lambda}(q,\varepsilon,\mathbf{Q})$ be the specialization of $\dot{\mathfrak{g}}_{\boldsymbol\lambda}=\dot{\mathfrak{g}}_{\boldsymbol\lambda}(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})$ at $(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})=(\varepsilon,q,\mathbf{Q})$.*

As the scalars $\dot{\mathfrak{g}}_{\boldsymbol\lambda}$ are central to all of our main results it is important to have a closed formula for them. Set $\sqrt{\boldsymbol\lambda}=(\boldsymbol\lambda^{[1]},\dots,\boldsymbol\lambda^{[\mathsf{o}_{\boldsymbol\lambda}]})$. Abusing notation slightly, $\boldsymbol\lambda=(\sqrt{\boldsymbol\lambda},\dots,\sqrt{\boldsymbol\lambda})$, where $\sqrt{\boldsymbol\lambda}$ is repeated $p_{\boldsymbol\lambda}$ times. Recall from before Corollary 3.9 that if $(i,j,s)\in[\boldsymbol\lambda]$ and $1\le t\le r$ then $h_{ij}^{\boldsymbol\lambda}(s,t)=\dot\varepsilon^{p_s-p_t}\dot q^{h_{ij}(\lambda^{(s)},\lambda^{(t)})}\dot Q_{d_s}\dot Q_{d_t}^{-1}$.

Proposition 3.40. *Suppose that $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$, and set $n_{\boldsymbol\lambda}=n/p_{\boldsymbol\lambda}$. Then there exists $k\in\mathbb{Z}$ such that*

$$\dot{\mathfrak{g}}_{\boldsymbol\lambda}=\dot\varepsilon^{\alpha(\boldsymbol\lambda)+k\mathsf{o}_{\boldsymbol\lambda}}\dot q^{\gamma_{\mathbf{b}}(\sqrt{\boldsymbol\lambda})}(\dot Q_1\dots\dot Q_d)^{n_{\boldsymbol\lambda}(p-1)}\prod_{(i,j,s)\in[\sqrt{\boldsymbol\lambda}]}\ \prod_{1\le t\le d\mathsf{o}_{\boldsymbol\lambda}}\ \prod_{\substack{0\le a<p_{\boldsymbol\lambda}\\ a\ne0\text{ if }p_t=p_s}}(\dot\varepsilon^{a\mathsf{o}_{\boldsymbol\lambda}}h_{ij}^{\boldsymbol\lambda}(s,t)-1),$$

where $\gamma_{\mathbf{b}}(\sqrt{\boldsymbol\lambda})=(\ell(w_{\mathbf{b}})+\sum_{a=1}^p\beta(\overrightarrow{\boldsymbol\lambda^{[a]}})-\beta(\overline{\boldsymbol\lambda}))/p_{\boldsymbol\lambda}$ and $\alpha(\boldsymbol\lambda)=\frac12 n_{\boldsymbol\lambda}(rp-d\mathsf{o}_{\boldsymbol\lambda})-d\alpha(\mathbf{b})/p_{\boldsymbol\lambda}$ are both integers.

Proof. First observe that $n_{\boldsymbol\lambda}=n/p_{\boldsymbol\lambda}=|\sqrt{\boldsymbol\lambda}|\in\mathbb{N}$. By Theorem 3.37, $\dot{\mathfrak{g}}_{\boldsymbol\lambda}^{p_{\boldsymbol\lambda}}=\dot\varepsilon^{\frac12 d\mathsf{o}_{\boldsymbol\lambda}n(p_{\boldsymbol\lambda}-1)}\dot{\mathfrak{f}}_{\boldsymbol\lambda}$. Therefore, by Corollary 3.9, $\dot{\mathfrak{g}}_{\boldsymbol\lambda}^{p_{\boldsymbol\lambda}}$ is equal to

$$\dot\varepsilon^{\frac12 d\mathsf{o}_{\boldsymbol\lambda}n(p_{\boldsymbol\lambda}-1)}\dot{\mathfrak{f}}_{\boldsymbol\lambda}=\dot q^{p_{\boldsymbol\lambda}\gamma_{\mathbf{b}}(\sqrt{\boldsymbol\lambda})}\dot\varepsilon^{p_{\boldsymbol\lambda}\alpha(\boldsymbol\lambda)}(\dot Q_1\dots\dot Q_d)^{n(p-1)}\prod_{(i,j,s)\in[\boldsymbol\lambda]}\ \prod_{\substack{1\le t\le r\\ p_t\ne p_s}}(h_{ij}^{\boldsymbol\lambda}(s,t)-1).$$

Observe that, because $\boldsymbol{\lambda} = \boldsymbol{\lambda}\langle o_{\boldsymbol{\lambda}}\rangle$, if $1 \le s,t \le d o_{\boldsymbol{\lambda}}$ and $0 \le a,b < p_{\boldsymbol{\lambda}}$ then $(i,j,s+a o_{\boldsymbol{\lambda}}) \in [\boldsymbol{\lambda}]$ and $h_{ij}^{\boldsymbol{\lambda}}(s,t) = \dot{\varepsilon}^{(a-b)o_{\boldsymbol{\lambda}}} h_{ij}^{\boldsymbol{\lambda}}(s+a o_{\boldsymbol{\lambda}}, t+b o_{\boldsymbol{\lambda}})$. Therefore,

$$\prod_{(i,j,s)\in[\boldsymbol{\lambda}]} \prod_{\substack{1\le t\le r\\ p_t\ne p_s}} (h_{ij}^{\boldsymbol{\lambda}}(s,t)-1) = \prod_{(i,j,s)\in[\sqrt{\boldsymbol{\lambda}}]} \prod_{0\le a<p_{\boldsymbol{\lambda}}} \prod_{1\le t\le d o_{\boldsymbol{\lambda}}} \prod_{\substack{0\le b<p_{\boldsymbol{\lambda}}\\ p_t+b o_{\boldsymbol{\lambda}}\ne p_s+a o_{\boldsymbol{\lambda}}}} (\dot{\varepsilon}^{(a-b)o_{\boldsymbol{\lambda}}} h_{ij}^{\boldsymbol{\lambda}}(s,t)-1).$$

Now, in the right hand products $1 \le s,t \le d o_{\boldsymbol{\lambda}}$, so $p_t + b o_{\boldsymbol{\lambda}} = p_s + a o_{\boldsymbol{\lambda}}$ if and only if $p_t = p_s$ and $a = b$. Therefore, the last equation becomes

$$\prod_{(i,j,s)\in[\boldsymbol{\lambda}]} \prod_{\substack{1\le t\le r\\ p_t\ne p_s}} (h_{ij}^{\boldsymbol{\lambda}}(s,t)-1) = \prod_{(i,j,s)\in[\sqrt{\boldsymbol{\lambda}}]} \prod_{1\le t\le d o_{\boldsymbol{\lambda}}} \prod_{\substack{0\le a<p_{\boldsymbol{\lambda}}\\ a\ne 0 \text{ if } p_t=p_s}} (\dot{\varepsilon}^{a o_{\boldsymbol{\lambda}}} h_{ij}^{\boldsymbol{\lambda}}(s,t)-1)^{p_{\boldsymbol{\lambda}}}.$$

Taking $p_{\boldsymbol{\lambda}}$th roots, the formula for $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}$ in the statement of the Proposition now follows. Note that this determines $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}$ only up to multiplication by $\dot{\varepsilon}^{k o_{\boldsymbol{\lambda}}}$, a $p_{\boldsymbol{\lambda}}$th root of unity, for some $k \in \mathbb{Z}$.

Finally, since $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}} \in \mathcal{A}$ by Theorem 3.37, it follows that $\gamma_{\mathbf{b}}(\sqrt{\boldsymbol{\lambda}})$ and $\alpha(\boldsymbol{\lambda})$ are both integers. We remark that it is not difficult to show this directly using just the definitions above. We leave this as an exercise for the reader. □

Remark 3.41. Proposition 3.40 determines $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}$ up to a $p_{\boldsymbol{\lambda}}$th root of unity $\dot{\varepsilon}^{k o_{\boldsymbol{\lambda}}}$. For the rest of this paper we make a fixed but arbitrary choice of the root of unity in Proposition 3.40. For the sake of definiteness, we take $k = 0$ and set

$$\dot{\mathfrak{g}}_{\boldsymbol{\lambda}} = \dot{\varepsilon}^{\alpha(\boldsymbol{\lambda})} \dot{q}^{\gamma_{\mathbf{b}}(\sqrt{\boldsymbol{\lambda}})} (\dot{Q}_1 \dots \dot{Q}_d)^{n_{\boldsymbol{\lambda}}(p-1)} \prod_{(i,j,s)\in[\sqrt{\boldsymbol{\lambda}}]} \prod_{1\le t\le d o_{\boldsymbol{\lambda}}} \prod_{\substack{0\le a<p_{\boldsymbol{\lambda}}\\ a\ne 0 \text{ if } p_t=p_s}} (\dot{\varepsilon}^{a o_{\boldsymbol{\lambda}}} h_{ij}^{\boldsymbol{\lambda}}(s,t)-1).$$

The formula for the splittable decomposition numbers in Theorem D, and all of the results which follow (for example, Definition 3.47), are relative to the choice of scalar $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}$. The reader can check that any other choice works equally well.

Theorem 3.37 shows that $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}} \in \mathcal{A}$ and, together with Proposition 3.4, this implies that the specialisation $\mathfrak{g}_{\boldsymbol{\lambda}} = \dot{\mathfrak{g}}_{\boldsymbol{\lambda}}(\varepsilon, q, \mathbf{Q})$ is well-defined and non-zero whenever $\mathbf{Q}$ is (ε, q)-separated. Proposition 3.40 immediately implies the following stronger characterisation of $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}$.

Corollary 3.42. *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}_{p,n}$. Then $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}} \in \mathbb{Z}[\dot{\varepsilon}, \dot{q}^{\pm 1}, \dot{Q}_1^{\pm 1}, \dots, \dot{Q}_d^{\pm 1}]$. Moreover, $\mathfrak{g}_{\boldsymbol{\lambda}} \ne 0$ whenever $\mathbf{Q}$ is (ε, q)-separated.*

The last two results follow directly from Corollary 3.9. In particular, they do not need the machinery developed in this section. The main results of this section are really Proposition 3.34 and Theorem 3.37 which connect the polynomials $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}$ with the representation theory of $\mathscr{H}_{r,p,n}$ via the shifting homomorphisms.

3.7. **Specht modules for $\mathscr{H}_{r,p,n}$.** Theorem 3.37 shows that $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}$ is a $p_{\boldsymbol{\lambda}}$th root of $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$. This implies that the endomorphism of $S(\boldsymbol{\lambda})$ induced by multiplication by $z_{\mathbf{b}}$ is a $p_{\boldsymbol{\lambda}}$th power of a 'simpler' endomorphism $\theta_{\boldsymbol{\lambda}}$. In this section we show that as a $\mathscr{H}_{r,p,n}$-module the Specht module decomposes into a direct sum of $\theta_{\boldsymbol{\lambda}}$-eigenspaces each of which is an $\mathscr{H}_{r,p,n}$-module. These eigenspaces are analogues of Specht modules for $\mathscr{H}_{r,p,n}$ and they will allow us to construct all of irreducible $\mathscr{H}_{r,p,n}$-modules.

Lemma 3.43. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and that $\mathbf{b} = \mathbf{b}\langle m\rangle$, for some $1 \le m \le p$ with m dividing p. Let $l = p/m$. Then $\theta'_{0,m} = \varepsilon^{dmnt/l} \sigma^{tm} \circ \theta'_{t,m} \circ \sigma^{-tm}$, for $0 \le t < l$.*

Proof. We first show that $\theta'_{t,m} = \varepsilon^{dmn/l}\sigma^m \circ \theta'_{t+1,m} \circ \sigma^{-m}$ whenever $0 \le t < l$. By construction both maps belong to $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}^{(tm)}, V_{\mathbf{b}}^{(tm+m)})$. By Lemma 3.18, $\sigma^m(Y_{t+1,m}) = \varepsilon^{-dmn/l}Y_{t,m}$. Consequently, if $v \in V_{\mathbf{b}}^{(tm)}$ then

$$\big(\sigma^m \circ \theta'_{t+1,m} \circ \sigma^{-m}\big)(v) = \sigma^m\big(Y_{t+1,m}\sigma^{-m}(v)\big) = \varepsilon^{-dmn/l}Y_{t,m}v = \varepsilon^{-dmn/l}\theta'_{t,m}(v)$$

Hence, $\theta'_{t,m} = \varepsilon^{dmn/l}\sigma^m \circ \theta'_{t+1,m} \circ \sigma^{-m}$ as claimed. Therefore, if $0 \le t < l$ then $\theta'_{0,m} = \varepsilon^{dmnt/l}\sigma^{tm} \circ \theta'_{t,m} \circ \sigma^{-tm}$ by induction on t. □

By Lemma 3.29, we have that $\theta_{t,m} = \sigma^m \circ \theta'_{t,m} \in \operatorname{End}_{\mathscr{H}_{r,p/m,n}}\big(V_{\mathbf{b}}^{(mt)}\big)$, for $0 \le t < p/m$. In particular, $\theta_{0,m} \in \operatorname{End}_{\mathscr{H}_{r,p/m,n}}\big(V_{\mathbf{b}}\big)$.

Lemma 3.44. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and that $\mathbf{b} = \mathbf{b}\langle m\rangle$, for some $1 \le m \le p$ with m dividing p. Let $l = p/m$. Then*

$$(\theta_{0,m})^l(v) = \varepsilon^{\frac{1}{2}dmn(l-1)}z_{\mathbf{b}} \cdot v,$$

for all $v \in V_{\mathbf{b}}$. That is, $\big(\theta_{0,m}\big)^l = \varepsilon^{\frac{1}{2}dmn(l-1)}z_{\mathbf{b}}$ as elements of $\operatorname{End}_{\mathscr{H}_{r,n}}\big(V_{\mathbf{b}}\big)$.

Proof. By Lemma 3.43, $\theta'_{0,m} = \varepsilon^{dmnt/l}\sigma^{tm} \circ \theta'_{t,m} \circ \sigma^{-tm}$ for $1 \le t < l$. Therefore,

$$\begin{aligned}\big(\theta_{0,m}\big)^l &= \big(\sigma^m \circ \theta'_{0,m}\big) \circ \big(\sigma^m \circ \theta'_{0,m}\big) \circ \cdots \circ \big(\sigma^m \circ \theta'_{0,m}\big)\\ &= \sigma^m \circ \varepsilon^{dmn(l-1)/l}\sigma^{(l-1)m} \circ \theta'_{l-1,m} \circ \sigma^{(1-l)m} \circ \sigma^m \circ \varepsilon^{dmn(l-2)/l}\sigma^{(l-2)m}\\ &\qquad \circ \theta'_{l-2,m} \circ \qquad \cdots \circ \sigma^m \circ \varepsilon^{dmn/l}\sigma^m \circ \theta'_{1,m} \circ \sigma^{-m} \circ \sigma^m \circ \theta'_{0,m}\\ &= \varepsilon^{\frac{1}{2}dmn(l-1)}\theta'_{l,m}\theta'_{l-1,m} \circ \cdots \circ \theta'_{0,m},\end{aligned}$$

since $\sigma^{lm} = \sigma^p$ is the identity map on $\mathscr{H}_{r,n}$. By Lemma 2.25 and the definitions, if $v \in V_{\mathbf{b}}$ then $\big(\theta'_{l-1,m} \circ \theta'_{l-2,m} \circ \cdots \circ \theta'_{0,m}\big)(v) = Y_p \dots Y_1 v = z_{\mathbf{b}} \cdot v$, so the result follows. □

Given $k \in \mathbb{Z}$ and a sequence $\mathbf{a} = (a_1, a_2, \dots, a_m)$ define $\mathbf{a}\langle k\rangle = (a_{k+1}, a_{k+2}, \dots, a_{k+m})$, where we set $a_{i+jm} = a_i$ whenever $j \in \mathbb{Z}$ and $1 \le i \le m$. Now define $\mathsf{o}_m(\mathbf{a}) = \min\{\, k \ge 1 \mid \mathbf{a}\langle k\rangle = \mathbf{a}\,\}$. In particular, i f $\mathbf{b} \in \mathscr{C}_{p,n}$ and $\boldsymbol{\lambda} = (\lambda^{[1]}, \dots, \lambda^{[p]}) \in \mathscr{P}_{d,\mathbf{b}}$ then this defines integers $\mathsf{o}_p(\mathbf{b})$ and $\mathsf{o}_p(\boldsymbol{\lambda})$. By definition, $o_p(\mathbf{b})$ and $\mathsf{o}_p(\boldsymbol{\lambda})$ both divide p, so $o^p(\mathbf{b})$ and $o^p(\boldsymbol{\lambda})$ are both integers. Further, $\mathsf{o}_p(\mathbf{b})$ divides $\mathsf{o}_p(\boldsymbol{\lambda})$.

For convenience, set $\mathsf{o}_{\boldsymbol{\lambda}} = \mathsf{o}_p(\boldsymbol{\lambda})$, $p_{\boldsymbol{\lambda}} = p/\mathsf{o}_{\boldsymbol{\lambda}}$, $\mathsf{o}_{\mathbf{b}} = \mathsf{o}_p(\mathbf{b})$ and $p_{\mathbf{b}} = p/\mathsf{o}_{\mathbf{b}}$. The definition of $\mathsf{o}_{\boldsymbol{\lambda}}$ and $p_{\boldsymbol{\lambda}}$ agree with those given in the introduction.

Definition 3.45. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Let $\theta_{\boldsymbol{\lambda}}$ be the restriction of $\theta_{0,\mathsf{o}_{\boldsymbol{\lambda}}}$ to $S(\boldsymbol{\lambda})$.*

As in Lemma 3.29, the image of $\theta_{\boldsymbol{\lambda}}$ is contained in $S(\boldsymbol{\lambda})$ so we can consider $\theta_{\boldsymbol{\lambda}}$ to be an $\mathscr{H}_{r,p_{\boldsymbol{\lambda}},n}$-module endomorphism of $S(\boldsymbol{\lambda})$. Recall the scalar $\mathfrak{g}_{\boldsymbol{\lambda}} = \dot{\mathfrak{g}}_{\boldsymbol{\lambda}}(\varepsilon, q, \mathbf{Q})$ from Definition 3.39 and Remark 3.41.

Corollary 3.46. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Then*

$$\big(\theta_{\boldsymbol{\lambda}}\big)^{p_{\boldsymbol{\lambda}}} = \mathfrak{g}_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}}1_{S(\boldsymbol{\lambda})},$$

where $1_{S(\boldsymbol{\lambda})}$ is the identity map on $S(\boldsymbol{\lambda})$.

Proof. Proposition 3.4 and Lemma 3.44 show that $(\theta_{\boldsymbol{\lambda}})^{p_{\boldsymbol{\lambda}}} = \varepsilon^{\frac{1}{2}dn\mathsf{o}_{\boldsymbol{\lambda}}(p_{\boldsymbol{\lambda}}-1)}\mathfrak{f}_{\boldsymbol{\lambda}}1_{S(\boldsymbol{\lambda})}$. Now apply Theorem 3.37. □

Definition 3.47. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$, $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ and $1 \le t \le p_{\boldsymbol{\lambda}}$. Define*

$$S_t^{\boldsymbol{\lambda}} = \{\, x \in S(\boldsymbol{\lambda}) \mid \theta_{\boldsymbol{\lambda}}(x) = \varepsilon^{to_{\boldsymbol{\lambda}}} \mathfrak{g}_{\boldsymbol{\lambda}} x \,\} = \ker\big(\theta_{\boldsymbol{\lambda}} - \varepsilon^{to_{\boldsymbol{\lambda}}} \mathfrak{g}_{\boldsymbol{\lambda}} 1_{S(\boldsymbol{\lambda})}\big).$$

Set $\pi_t^{\boldsymbol{\lambda}} = \prod_{1 \le s \le p_{\boldsymbol{\lambda}}, s \ne t} \big(\theta_{\boldsymbol{\lambda}} - \varepsilon^{so_{\boldsymbol{\lambda}}} \mathfrak{g}_{\boldsymbol{\lambda}}\big)$, so that $\pi_t^{\boldsymbol{\lambda}} \in \operatorname{End}_{\mathscr{H}_{r,p_{\boldsymbol{\lambda}},n}}\big(S(\boldsymbol{\lambda})\big)$.

By definition, $S_t^{\boldsymbol{\lambda}}$ is an $\mathscr{H}_{r,p_{\boldsymbol{\lambda}},n}$-submodule of $S(\boldsymbol{\lambda})$, for $1 \le t \le p_{\boldsymbol{\lambda}}$. By restriction, we consider $S_t^{\boldsymbol{\lambda}}$ to be an $\mathscr{H}_{r,p,n}$-module. Recall that τ is the automorphism of $\mathscr{H}_{r,n}$ given by $\tau(h) = T_0^{-1} h T_0$, for $h \in \mathscr{H}_{r,n}$.

Theorem 3.48. *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}_{p,n}$, and $1 \le t \le p_{\boldsymbol{\lambda}}$. Then*

a) *$S_t^{\boldsymbol{\lambda}} T_0 = S_{t+1}^{\boldsymbol{\lambda}}$. Equivalently, $\big(S_{t+1}^{\boldsymbol{\lambda}}\big)^{\tau} \cong S_t^{\boldsymbol{\lambda}}$.*
b) *$S_t^{\boldsymbol{\lambda}} = \pi_t^{\boldsymbol{\lambda}}\big(S(\boldsymbol{\lambda})\big)$;*
c) *$S(\boldsymbol{\lambda})\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong S_1^{\boldsymbol{\lambda}} \oplus \cdots \oplus S_{p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}}$;*
d) *$\dim S_t^{\boldsymbol{\lambda}} = \frac{1}{p_{\boldsymbol{\lambda}}} \dim S(\boldsymbol{\lambda})$;*
e) *$S_t^{\boldsymbol{\lambda}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong S(\boldsymbol{\lambda}) \oplus S(\boldsymbol{\lambda})^{\sigma} \oplus \cdots \oplus S(\boldsymbol{\lambda})^{\sigma^{(o_{\boldsymbol{\lambda}}-1)}}$.*

Proof. Suppose that $x \in S_t^{\boldsymbol{\lambda}}$ and let $m = o_{\boldsymbol{\lambda}}$. By definition,

$$\theta_{\boldsymbol{\lambda}}(xT_0) = \big(\sigma^m \circ \theta'_{0,m}\big)(xT_0) = \sigma^m\big(\theta'_{0,o_p(\boldsymbol{\lambda})}(x)T_0\big),$$

since $\theta'_{0,m}$ is an $\mathscr{H}_{r,n}$-module homomorphism. Therefore,

$$\theta_{\boldsymbol{\lambda}}(xT_0) = \theta_{\boldsymbol{\lambda}}(x)\sigma^m(T_0) = \varepsilon^{(t+1)m} \mathfrak{g}_{\boldsymbol{\lambda}} x T_0.$$

Hence, $xT_0 \in S_{t+1}^{\boldsymbol{\lambda}}$, proving the first half of (a). That $S_{t+1}^{\boldsymbol{\lambda}} \cong \big(S_t^{\boldsymbol{\lambda}}\big)^{\tau}$ is now immediate because if $x \in S_{t+1}^{\boldsymbol{\lambda}}$ then $x = x'T_0$ for some $x' \in S_t^{\boldsymbol{\lambda}}$. Therefore, if $h \in \mathscr{H}_{r,n}$ then $xh = x'T_0h = x'\tau(h)T_0$. Hence, we have proved (a).

By Corollary 3.46, the map $\theta_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}} - \mathfrak{g}_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}}$ kills every element of $S(\boldsymbol{\lambda})$. Thus, on $S(\boldsymbol{\lambda})$ we have

$$0 = \theta_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}} - \mathfrak{g}_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}} = \prod_{1 \le s \le p_{\boldsymbol{\lambda}}} \big(\theta_{\boldsymbol{\lambda}} - \varepsilon^{so_{\boldsymbol{\lambda}}} \mathfrak{g}_{\boldsymbol{\lambda}}\big) = \pi_t^{\boldsymbol{\lambda}} \circ \big(\theta_{\boldsymbol{\lambda}} - \varepsilon^{to_{\boldsymbol{\lambda}}} \mathfrak{g}_{\boldsymbol{\lambda}}\big).$$

Hence, the image of $\pi_t^{\boldsymbol{\lambda}}$ is contained in $S_t^{\boldsymbol{\lambda}}$ and $\ker \pi_t^{\boldsymbol{\lambda}} = \sum_{s \ne t} S_s^{\boldsymbol{\lambda}}$. Note that the assumption $\mathfrak{f}_{\boldsymbol{\lambda}}$ is invertible in R implies that $\mathfrak{g}_{\boldsymbol{\lambda}}$ is also invertible in R. If $x \in S_t^{\boldsymbol{\lambda}}$ then $\pi_t^{\boldsymbol{\lambda}}(x) = \alpha_t x$, where $\alpha_t = \mathfrak{g}_{\boldsymbol{\lambda}} \prod_{s \ne t}(\varepsilon^{to_{\boldsymbol{\lambda}}} - \varepsilon^{so_{\boldsymbol{\lambda}}})$ is invertible in R. It follows that if we set $\hat{\pi}_s^{\boldsymbol{\lambda}} = \frac{1}{\alpha_s} \pi_s^{\boldsymbol{\lambda}}$ then

$$1_{S(\boldsymbol{\lambda})} = \hat{\pi}_1^{\boldsymbol{\lambda}} + \hat{\pi}_2^{\boldsymbol{\lambda}} + \cdots + \hat{\pi}_{p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}},$$

and $\hat{\pi}_t^{\boldsymbol{\lambda}}$ is the projection map from $S(\boldsymbol{\lambda})$ onto $S_t^{\boldsymbol{\lambda}}$. Hence, (b) and (c) now follow. Moreover, since $\dim S_t^{\boldsymbol{\lambda}} = \dim S_{t+1}^{\boldsymbol{\lambda}}$ by (a), we obtain (d) from (c).

It remains then to prove (e). First observe that by part (a),

$$S_t^{\boldsymbol{\lambda}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong (S_{t+1}^{\boldsymbol{\lambda}})^{\tau}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong \big(S_{t+1}^{\boldsymbol{\lambda}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\big)^{\tau} \cong S_{t+1}^{\boldsymbol{\lambda}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}.$$

Therefore, $S_1^{\boldsymbol\lambda}\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}} \cong \dots \cong S_{p_{\boldsymbol\lambda}}^{\boldsymbol\lambda}\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}$. Hence, using part (c) — which we have already proved — and applying Corollary 3.17, we see that

$$\Big(S_t^{\boldsymbol\lambda}\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\Big)^{\oplus p_{\boldsymbol\lambda}} \cong \big(S_1^{\boldsymbol\lambda}\oplus\cdots\oplus S_{p_{\boldsymbol\lambda}}^{\boldsymbol\lambda}\big)\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}} \cong S(\boldsymbol\lambda)\downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}} \cong \bigoplus_{j=0}^{p-1} S(\boldsymbol\lambda)^{\sigma^j}$$
$$\cong \Big(\bigoplus_{j=0}^{\mathsf{o}_{\boldsymbol\lambda}-1} S(\boldsymbol\lambda)^{\sigma^j}\Big)^{\oplus p_{\boldsymbol\lambda}},$$

where the last isomorphism follows because $S(\boldsymbol\lambda)^{\sigma^t}\cong S(\boldsymbol\lambda\langle -t\rangle)$ by Proposition 3.24. Applying the Krull–Schmidt theorem we deduce

$$S_t^{\boldsymbol\lambda}\uparrow^{\mathscr{H}_{r,n}} \cong S(\boldsymbol\lambda)\oplus S(\boldsymbol\lambda)^{\sigma}\oplus\cdots\oplus S(\boldsymbol\lambda)^{\sigma^{(\mathsf{o}_{\boldsymbol\lambda}-1)}},$$

proving (e). This completes the proof of Theorem 3.48. □

As in the introduction, let $\sim_\sigma$ be the equivalence relation on $\mathscr{P}_{r,n}$ where $\boldsymbol\mu\sim_\sigma\boldsymbol\lambda$ whenever $\boldsymbol\lambda=\boldsymbol\mu\langle m\rangle$, for some $m\in\mathbb{Z}$. Let $\mathscr{P}_{r,n}^\sigma$ be the set of $\sim_\sigma$-equivalence classes in $\mathscr{P}_{r,n}$. By Proposition 3.24, the set $\mathscr{K}_{r,n}$ of Kleshchev multipartitions is closed under $\sim_\sigma$-equivalence. Let $\mathscr{K}_{r,n}^\sigma$ be the set of $\sim_\sigma$-equivalence classes of Kleshchev multipartitions. We will abuse notation and think of the elements of $\mathscr{P}_{r,n}^\sigma$ as multipartitions so that when we write $\boldsymbol\mu\in\mathscr{P}_{r,n}^\sigma$ we will really mean that $\boldsymbol\mu$ is a representative of an equivalence class in $\mathscr{P}_{r,n}^\sigma$. Similarly, $\boldsymbol\mu\in\mathscr{K}_{r,n}^\sigma$ means that $\boldsymbol\mu$ is a representative for an equivalence class in $\mathscr{K}_{r,n}^\sigma$.

Let $R=K$ be a field. We call the modules $\{\,S_i^{\boldsymbol\lambda}\mid\boldsymbol\lambda\in\mathscr{P}_{r,n}^\sigma \text{ and } 1\le i\le p_{\boldsymbol\lambda}\,\}$ the **Specht modules** of $\mathscr{H}_{r,p,n}$. Using these modules we can now construct the irreducible $\mathscr{H}_{r,p,n}$-modules.

Definition 3.49. *Suppose that $\boldsymbol\lambda\in\mathscr{K}_{r,n}$ and $1\le t\le p_{\boldsymbol\lambda}$. Define $D_t^{\boldsymbol\lambda}=\operatorname{Head}(S_t^{\boldsymbol\lambda})$.*

Although this is not clear from the definition, the module $D_i^{\boldsymbol\lambda}$ is irreducible when $\boldsymbol\lambda\in\mathscr{K}_{r,n}$ and, moreover every irreducible $\mathscr{H}_{r,p,n}$-module arises in this way.

This following result establishes of Theorem C from the introduction and, in fact, proves quite a bit more.

Theorem 3.50. *Suppose that $\mathbf{Q}$ is (ε,q)-separated over the field K. Let $\boldsymbol\lambda\in\mathscr{K}_{r,n}$. Then:*

a) *The module $D_i^{\boldsymbol\lambda}=\operatorname{Head}(S_i^{\boldsymbol\lambda})$ is an irreducible $\mathscr{H}_{r,p,n}$-module, for $1\le i\le p_{\boldsymbol\lambda}$. Moreover, $(D_{i+1}^{\boldsymbol\lambda})^\tau\cong D_i^{\boldsymbol\lambda}$, for $1\le i\le p_{\boldsymbol\lambda}$.*
b) *If $1\le i,j\le p_{\boldsymbol\lambda}$ then $[S_i^{\boldsymbol\lambda}:D_j^{\boldsymbol\lambda}]=\delta_{ij}$.*
c) *The integer $p_{\boldsymbol\lambda}$ is the smallest positive integer such that $D_i^{\boldsymbol\lambda}\cong\big(D_i^{\boldsymbol\lambda}\big)^{\tau^{p_{\boldsymbol\lambda}}}$.*
d) *The integer $\mathsf{o}_{\boldsymbol\lambda}$ is the smallest positive integer such that $D(\boldsymbol\lambda)\cong D(\boldsymbol\lambda)^{\sigma^{\mathsf{o}_{\boldsymbol\lambda}}}$.*
e) *$(D_i^{\boldsymbol\lambda})\uparrow^{\mathscr{H}_{r,n}}\cong D(\boldsymbol\lambda)\oplus D(\boldsymbol\lambda)^\sigma\oplus\cdots\oplus D(\boldsymbol\lambda)^{\sigma^{\mathsf{o}_{\boldsymbol\lambda}-1}}$ and $D(\boldsymbol\lambda)\downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\cong D_i^{\boldsymbol\lambda}\oplus(D_i^{\boldsymbol\lambda})^\tau\oplus\cdots\oplus\big(D_i^{\boldsymbol\lambda}\big)^{\tau^{p_{\boldsymbol\lambda}-1}}$.*

Furthermore, the Hecke algebra $\mathscr{H}_{r,p,n}$ is split over K and

$$\{\,D_i^{\boldsymbol\mu}\mid\boldsymbol\mu\in\mathscr{K}_{r,n}^\sigma \text{ and } 1\le i\le p_{\boldsymbol\mu}\,\}$$

is a complete set of pairwise non-isomorphic absolutely irreducible $\mathscr{H}_{r,p,n}$-modules.

Proof. By Proposition 3.24, $D(\boldsymbol{\lambda})^\sigma \cong D(\boldsymbol{\lambda}\langle -1\rangle)$, so it is clear that $o_{\boldsymbol{\lambda}}$ is the smallest positive integer such that $D(\boldsymbol{\lambda}) \cong D(\boldsymbol{\lambda})^{\sigma^{o_{\boldsymbol{\lambda}}}}$. Similarly, once we know that $D^{\boldsymbol{\lambda}}_i = \operatorname{Head}(S^{\boldsymbol{\lambda}}_i)$ is irreducible then $(D^{\boldsymbol{\lambda}}_{i+1})^\tau \cong D^{\boldsymbol{\lambda}}_i$ by Theorem 3.48(a) since twisting by τ induces an exact functor on Mod-$\mathscr{H}_{r,p,n}$.

For the other statements, we first consider the case where $K = \overline{K}$ is algebraically closed so that $\mathscr{H}^{\overline{K}}_{r,p,n}$ splits over $\overline{K}$. The algebra $\mathscr{H}_{r,n}$ is cellular over any ring and so, in particular, it is split over K. Therefore, if $\boldsymbol{\mu} \in \mathscr{K}^\sigma_{r,n}$ then $D^{\overline{K}}(\boldsymbol{\mu}) = D(\boldsymbol{\mu}) \otimes_K \overline{K}$. Fix an irreducible $\mathscr{H}^{\overline{K}}_{r,p,n}$-submodule $D^{\boldsymbol{\mu}}_{\overline{K}}$ of $D^{\overline{K}}(\boldsymbol{\mu})$. By Lemma 3.11, the integer $p_{\boldsymbol{\lambda}}$ is the smallest positive integer such that $D^{\boldsymbol{\lambda}}_{\overline{K}} \cong (D^{\boldsymbol{\lambda}}_{\overline{K}})^{\tau^{p_{\boldsymbol{\lambda}}}}$ and, further,

$$D^{\overline{K}}(\boldsymbol{\lambda})\downarrow_{\mathscr{H}^{\overline{K}}_{r,p,n}} \cong D^{\boldsymbol{\lambda}}_{\overline{K}} \oplus (D^{\boldsymbol{\lambda}}_{\overline{K}})^\tau \oplus \cdots \oplus (D^{\boldsymbol{\lambda}}_{\overline{K}})^{\tau^{p_{\boldsymbol{\lambda}}-1}}$$

and

$$D^{\boldsymbol{\lambda}}_{\overline{K}}\uparrow^{\mathscr{H}^{\overline{K}}_{r,n}} \cong D^{\overline{K}}(\boldsymbol{\lambda}) \oplus D^{\overline{K}}(\boldsymbol{\lambda})^\sigma \oplus \cdots \oplus D^{\overline{K}}(\boldsymbol{\lambda})^{\sigma^{o_{\boldsymbol{\lambda}}-1}}.$$

Moreover, $\{\, \big(D^{\boldsymbol{\mu}}_{\overline{K}}\big)^{\tau^i} \mid \boldsymbol{\mu} \in \mathscr{K}^\sigma_{r,n} \text{ and } 1 \le i \le p_{\boldsymbol{\mu}} \,\}$ is a complete set of pairwise non-isomorphic simple $\mathscr{H}^{\overline{K}}_{r,p,n}$-modules.

Suppose that $\boldsymbol{\mu} \in \mathscr{P}_{r,n}$ and let $S^{\boldsymbol{\mu}}_{\overline{K},i} = S^{\boldsymbol{\mu}}_i \otimes_K \overline{K}$, for $1 \le j \le p_{\boldsymbol{\mu}}$. We claim that $D^{\boldsymbol{\lambda}}_{\overline{K}} \cong \operatorname{Head}(S^{\boldsymbol{\mu}}_{\overline{K},i})$, for some i, if and only if $\boldsymbol{\lambda} \sim_\sigma \boldsymbol{\mu}$ and in this case i is uniquely determined. Using the restriction formula for $D^{\overline{K}}(\boldsymbol{\lambda})$ given above, Frobenius reciprocity and Theorem 3.48 we find that

$$\begin{aligned}
\bigoplus_{i=0}^{p_{\boldsymbol{\mu}}-1} \operatorname{Hom}_{\mathscr{H}^{\overline{K}}_{r,p,n}}\big(S^{\boldsymbol{\mu}}_{\overline{K},i}, D^{\boldsymbol{\lambda}}_{\overline{K}}\big) &\cong \operatorname{Hom}_{\mathscr{H}^{\overline{K}}_{r,p,n}}\big(S^{\overline{K}}(\boldsymbol{\mu})\downarrow_{\mathscr{H}^{\overline{K}}_{r,p,n}}, D^{\boldsymbol{\lambda}}_{\overline{K}}\big)\\
&\cong \operatorname{Hom}_{\mathscr{H}^{\overline{K}}_{r,n}}\big(S^{\overline{K}}(\boldsymbol{\mu}), D^{\boldsymbol{\lambda}}_{\overline{K}}\uparrow^{\mathscr{H}^{\overline{K}}_{r,n}}\big)\\
&\cong \bigoplus_{j=0}^{o_{\boldsymbol{\lambda}}-1} \operatorname{Hom}_{\mathscr{H}^{\overline{K}}_{r,n}}\big(S^{\overline{K}}(\boldsymbol{\mu}), D^{\overline{K}}(\boldsymbol{\lambda})^{\sigma^j}\big)\\
&\cong \begin{cases} \overline{K}, & \text{if } \boldsymbol{\mu} \sim_\sigma \boldsymbol{\lambda},\\ 0, & \text{otherwise,}\end{cases}
\end{aligned}$$

where the last line follows because $D^{\overline{K}}(\boldsymbol{\mu}) = \operatorname{Head}(S^{\overline{K}}(\boldsymbol{\mu}))$, by Lemma 3.2, and because $D^{\overline{K}}(\boldsymbol{\lambda})^{\sigma^j} \cong D^{\overline{K}}(\boldsymbol{\lambda}\langle -j\rangle)$ by Proposition 3.24. This proves our claim. Without loss of generality, we can take $\boldsymbol{\mu} = \boldsymbol{\lambda}$. Note that $\operatorname{Head} S^{\overline{K}}(\boldsymbol{\lambda}) = D^{\overline{K}}(\boldsymbol{\lambda})$ is simple. The above isomorphisms imply that $\operatorname{Head}(S^{\boldsymbol{\lambda}}_{\overline{K},i}) = D^{\boldsymbol{\lambda}}_{\overline{K},i} = D^{\boldsymbol{\lambda}}_{\overline{K}}$ is also simple. By Lemma 3.2, $[S^{\overline{K}}(\boldsymbol{\lambda}) : D^{\overline{K}}(\boldsymbol{\lambda})] = 1$ and $D^{\overline{K}}(\boldsymbol{\lambda})$ is the simple head of $S^{\overline{K}}(\boldsymbol{\lambda})$. By considering the restriction of the composition series of $S^{\overline{K}}(\boldsymbol{\lambda})$ to $\mathscr{H}_{r,p,n}$, it follows that $[S^{\boldsymbol{\lambda}}_{\overline{K},i} : D^{\boldsymbol{\lambda}}_{\overline{K},j}] = \delta_{ij}$. This proves all the statements in the Theorem when $K = \overline{K}$.

We now return to the general case where K is an arbitrary field. By the last paragraph, $S^{\boldsymbol{\lambda}}_{\overline{K},i} \cong S^{\boldsymbol{\lambda}}_i \otimes_K \overline{K}$ has a simple head, so that $D^{\boldsymbol{\lambda}}_i = \operatorname{Head}(S^{\boldsymbol{\lambda}}_i)$ is indecomposable. Therefore, $D^{\boldsymbol{\lambda}}_i$ is irreducible (since it is also semisimple).

To complete the proof of the Theorem we show that $D^{\boldsymbol{\lambda}}_i \otimes_K \overline{K} \cong D^{\boldsymbol{\lambda}}_{\overline{K},i}$. Let $l \ge 1$ be the minimal positive integer such that $(D^{\boldsymbol{\lambda}}_i)^{\tau^l} \cong D^{\boldsymbol{\lambda}}_i$. Then $l \ge p_{\boldsymbol{\lambda}}$ since $D^{\boldsymbol{\lambda}}_{\overline{K},i} \cong$

$\mathrm{Head}(D_i^{\boldsymbol\lambda}\otimes_K\overline{K})$. Similarly, $\dim_K D_i^{\boldsymbol\lambda}\ge\dim_{\overline{K}} D_{\overline{K},i}^{\boldsymbol\lambda}$. By [9, Proposition 11.16], there exists an integer $c\ge 1$ such that

$$D(\boldsymbol\lambda)\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\cong\Big(D_i^{\boldsymbol\lambda}\oplus(D_i^{\boldsymbol\lambda})^{\tau}\oplus\cdots\oplus\big(D_i^{\boldsymbol\lambda}\big)^{\tau^{l-1}}\Big)^{\oplus c}.$$

Taking dimensions, $\dim_K D(\boldsymbol\lambda)=cl\dim_K D_i^{\boldsymbol\lambda}$. Hence, comparing dimensions on both sides of the restriction formula for $D^{\overline{K}}(\boldsymbol\lambda)$ above shows that

$$cl\dim D_i^{\boldsymbol\lambda}=\dim_K D(\boldsymbol\lambda)=\dim_{\overline{K}} D^{\overline{K}}(\boldsymbol\lambda)=p_{\boldsymbol\lambda}\dim_{\overline{K}} D_{\overline{K}}^{\boldsymbol\lambda}\le p_{\boldsymbol\lambda}\dim_K D_i^{\boldsymbol\lambda}.$$

Since $l\ge p_{\boldsymbol\lambda}$ this forces $c=1$, $l=p_{\boldsymbol\lambda}$ and $\dim_K D_i^{\boldsymbol\lambda}=\dim_{\overline{K}} D_{\overline{K},i}^{\boldsymbol\lambda}$. Therefore, $D_{\overline{K},i}^{\boldsymbol\lambda}\cong D_i^{\boldsymbol\lambda}\otimes_K\overline{K}$, implying that $D_i^{\boldsymbol\lambda}$ is absolutely irreducible and hence that K is a splitting field for $\mathscr{H}_{r,p,n}$. All of the parts in the theorem now follow from the corresponding statements for $D_{\overline{K},i}^{\boldsymbol\lambda}$ using the isomorphism $D_{\overline{K},i}^{\boldsymbol\lambda}\cong D_i^{\boldsymbol\lambda}\otimes_K\overline{K}$. □

The algebra $\mathscr{H}_{r,n}(\mathbf{Q}^{\vee\varepsilon})$ is not necessarily semisimple when $d>1$. With a little more work it is possible to show that if $\mathbf{Q}$ is (ε,q)-separated over K then the following are equivalent:

a) $\mathscr{H}_{r,n}$ is (split) semisimple.
b) $\mathscr{H}_{r,p,n}$ is (split) semisimple.
c) $S_t^{\boldsymbol\lambda}=D_t^{\boldsymbol\lambda}$, for all $\boldsymbol\lambda\in\mathscr{P}_{r,n}$ and $1\le t\le p_{\boldsymbol\lambda}$.

We omit the details. If $d=1$ then it is known that $\mathscr{H}_{p,p,n}$ is semisimple if and only if $\langle\varepsilon\rangle\cap\langle q\rangle=\{1\}$ and $e>n$ [22, Theorem 5.9].

Extend the dominance order to $\mathscr{P}_{r,n}^{\sigma}\times\mathbb{Z}$ by defining $(\boldsymbol\lambda,j)\rhd(\boldsymbol\mu,i)$ if $\boldsymbol\lambda\rhd\boldsymbol\mu$. Let

$$\mathbf{D}_{\mathscr{H}_{r,p,n}}=\big([S_i^{\boldsymbol\lambda}:D_j^{\boldsymbol\mu}]\big)_{(\boldsymbol\lambda,i),(\boldsymbol\mu,j)}$$

be the **decomposition matrix** of $\mathscr{H}_{r,p,n}$, where $\boldsymbol\lambda\in\mathscr{P}_{r,n}^{\sigma}$, $\boldsymbol\mu\in\mathscr{K}_{r,n}^{\sigma}$, $1\le i\le p_{\boldsymbol\lambda}$ and $1\le j\le p_{\boldsymbol\mu}$, and where the rows and columns of $\mathbf{D}_{\mathscr{H}_{r,p,n}}$ are ordered in a way that is compatible with dominance.

Suppose that $\boldsymbol\lambda\in\mathscr{P}_{r,n}$, $\boldsymbol\mu\in\mathscr{K}_{r,n}^{\sigma}$ and $1\le i\le p_{\boldsymbol\lambda}$ and $1\le i\le p_{\boldsymbol\mu}$. If $\boldsymbol\lambda\ne\boldsymbol\mu$ then $[S_i^{\boldsymbol\lambda}:D_j^{\boldsymbol\mu}]\ne 0$ only if $(\boldsymbol\lambda,i)\rhd(\boldsymbol\mu,j)$ because, by Theorem 3.50 and Lemma 3.2,

$$[S_i^{\boldsymbol\lambda}:D_j^{\boldsymbol\mu}]\ne 0\quad\Longrightarrow\quad[S(\boldsymbol\lambda):D(\boldsymbol\mu)]\ne 0\quad\Longrightarrow\quad\boldsymbol\lambda\rhd\boldsymbol\mu.$$

On the other hand, $[S_i^{\boldsymbol\mu}:D_j^{\boldsymbol\mu}]=\delta_{ij}$ by Theorem 3.50. Hence, we have proved the following.

Corollary 3.51. *Suppose that $\mathbf{Q}$ is (ε,q)-separated over the field K. Then the decomposition matrix $\mathbf{D}_{\mathscr{H}_{r,p,n}}$ of $\mathscr{H}_{r,p,n}$ is unitriangular.*

Theorem 3.50 and Corollary 3.51 complete the proof of Theorem C from the introduction.

4. Cyclotomic Schur algebras and decomposition numbers

We have now constructed Specht modules and a complete set of simple modules for $\mathscr{H}_{r,p,n}$. In particular, we have proved Theorems B and C from the introduction. The key to proving Theorem C was the construction of the $\mathscr{H}_{r,p,n}$-endomorphism $\theta_{\boldsymbol\lambda}$ of the Specht module $S(\boldsymbol\lambda)$ in Definition 3.45.

In this chapter we compute the p-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$. To do this we first construct a new algebra $\mathscr{E}_d$ which is Morita equivalent to $\mathscr{H}_{r,p,n}$. This allows us to construct an analogue of the (cyclotomic) Schur algebra for $\mathscr{H}_{r,p,n}$.

The endomorphisms $\theta_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, lift to analogous elements $\vartheta_{\boldsymbol{\lambda}}$ of $\mathscr{S}_{r,p,n}$. Extending the arguments of [24], we compute the trace of $\vartheta_{\boldsymbol{\lambda}}$ on certain weight spaces (the twining characters). These trace functions give a system of linear equations which determine the p-splittable decomposition numbers of the three algebras $\mathscr{S}_{r,p,n}$, $\mathscr{E}_d$ and $\mathscr{H}_{r,p,n}$. This will complete the proofs of Theorems A and D from the introduction.

Many of the early results in this section hold over an integral domain, however, for convenience we work over a field $R=K$. We maintain our assumption that $\mathbf{Q}$ is (ε,q)-separated over K.

4.1. **A Morita equivalence for $\mathscr{H}_{r,p,n}$.** In this section we prove a new Morita equivalence theorem for the cyclotomic Hecke algebras $\mathscr{H}_{r,p,n}$ which is an analogue of Theorem 2.5. This equivalence (Corollary 4.5) is both a refinement of [25, Theorem A] and a generalization of the Morita equivalence theorem given by the first author for the Hecke algebras of type D [21].

Fix a composition $\mathbf{b}\in\mathscr{C}_{p,n}$ and set $\mathsf{o}_{\mathbf{b}}=\mathsf{o}_p(\mathbf{b})$ and $p_{\mathbf{b}}=p/\mathsf{o}_{\mathbf{b}}$. Mirroring Definition 3.45 define

$$\theta_{\mathbf{b}}=\theta_{0,\mathsf{o}_p(\mathbf{b})}.$$

Then $\theta_{\mathbf{b}}\in\operatorname{End}_{\mathscr{H}_{r,p_{\mathbf{b}},n}}(V_{\mathbf{b}})$ by Lemma 3.29 and $\theta_{\mathbf{b}}(v)=\sigma^{\mathsf{o}_{\mathbf{b}}}(Y_{0,\mathsf{o}_{\mathbf{b}}}v)$, for all $v\in V_{\mathbf{b}}$. In particular, $\theta_{\mathbf{b}}$ is an $\mathscr{H}_{r,p,n}$-endomorphism of $V_{\mathbf{b}}$.

The module $V_{\mathbf{b}}=v_{\mathbf{b}}\mathscr{H}_{r,n}$ is an $\mathscr{H}_{r,p,n}$-module by restriction. For simplicity we will usually write $V_{\mathbf{b}}$, instead of $V_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$, when we consider $V_{\mathbf{b}}$ as an $\mathscr{H}_{r,p,n}$-module.

Definition 4.1. *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$. Define $\mathscr{E}_{d,\mathbf{b}}=\operatorname{End}_{\mathscr{H}_{r,p,n}}\big(V_{\mathbf{b}}\big)$.*

Notice that $\mathscr{H}_{d,\mathbf{b}}$ is a subalgebra of $\mathscr{E}_{d,\mathbf{b}}$, by Proposition 2.23(a), and that $\theta_{\mathbf{b}}$ is an element of $\mathscr{E}_{d,\mathbf{b}}$ by the remarks above.

Theorem 4.2. *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$. Then, as an algebra, $\mathscr{E}_{d,\mathbf{b}}$ is generated by $\mathscr{H}_{d,\mathbf{b}}$ and the endomorphism $\theta_{\mathbf{b}}$. Moreover, if $\{\,x_i\mid i\in I\,\}$ is a K-basis of $\mathscr{H}_{d,\mathbf{b}}$ then $\{\,x_i\theta_{\mathbf{b}}^k\mid i\in I$ and $0\le k<p_{\mathbf{b}}\,\}$ is a K-basis of $\operatorname{End}_{\mathscr{H}_{r,p,n}}(V_{\mathbf{b}})$. In particular, $\dim\mathscr{E}_{d,\mathbf{b}}=p_{\mathbf{b}}\dim\mathscr{H}_{d,\mathbf{b}}$.*

Proof. We first compute the dimension of $\mathscr{E}_{d,\mathbf{b}}$. By Frobenius reciprocity,

$$\begin{aligned}\mathscr{E}_{d,\mathbf{b}}&=\operatorname{Hom}_{\mathscr{H}_{r,p,n}}(V_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}},V_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}})\cong\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}},V_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}})\\&\cong\bigoplus_{i=0}^{p-1}\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}},V_{\mathbf{b}}^{\sigma^i})\cong\bigoplus_{i=0}^{p-1}\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}},V_{\mathbf{b}\langle i\rangle}),\end{aligned}$$

where the third isomorphism is Corollary 3.17 and the fourth isomorphism follows because $V_{\mathbf{b}}^{\sigma^i}\cong V_{\mathbf{b}\langle -i\rangle}$ by Proposition 3.19. By [25, Proposition 2.13] if $\mathbf{b}\ne\mathbf{c}$ then $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}},V_{\mathbf{c}})=0$ because $V_{\mathbf{b}}$ and $V_{\mathbf{c}}$ belong to different blocks. Therefore, as vector spaces,

$$\mathscr{E}_{d,\mathbf{b}}\cong\bigoplus_{i=0}^{p_{\mathbf{b}}-1}\operatorname{Hom}_{\mathscr{H}_{r,n}}\big(V_{\mathbf{b}},V_{\mathbf{b}\langle i\mathsf{o}_{\mathbf{b}}\rangle}\big)\cong\bigoplus_{i=0}^{p_{\mathbf{b}}-1}\operatorname{Hom}_{\mathscr{H}_{r,n}}\big(V_{\mathbf{b}},V_{\mathbf{b}}\big)\cong\mathscr{H}_{d,\mathbf{b}}^{\oplus p_{\mathbf{b}}}$$

since $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}})\cong\mathscr{H}_{d,\mathbf{b}}$ by Proposition 2.23(a). Hence, $\dim\mathscr{E}_{d,\mathbf{b}}=p_{\mathbf{b}}\dim\mathscr{H}_{d,\mathbf{b}}$.

It remains to show that $\mathscr{H}_{d,\mathbf{b}}$ and $\theta_{\mathbf{b}}$ generate $\mathscr{E}_{d,\mathbf{b}}$ as a K-algebra. First observe that $\theta_{\mathbf{b}}$ is an invertible element of $\mathscr{E}_{d,\mathbf{b}}$ because $(\theta_{\mathbf{b}})^{p_{\mathbf{b}}}(v)=\varepsilon^{dn(p-\mathsf{o}_{\mathbf{b}})/2}z_{\mathbf{b}}\cdot v$

by Lemma 3.44, for $v \in V_{\mathbf{b}}$. Therefore, since $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}) \cong \mathscr{H}_{d,\mathbf{b}}$ by Proposition 2.23(a), it suffices to show that every element of $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}, V_{\mathbf{b}}^{\sigma^{io_{\mathbf{b}}}})$ corresponds to $\theta_{\mathbf{b}}^{-i}x$, for some x in $\mathscr{H}_{d,\mathbf{b}}$. Let π_j be the projection from $\mathscr{E}_{d,\mathbf{b}}$ to $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}, V_{\mathbf{b}}^{\sigma^{jo_{\mathbf{b}}}})$ under the vector space isomorphism above. Under Frobenius reciprocity the $\mathscr{H}_{r,p,n}$-endomorphism

$$\theta_{\mathbf{b}}^{-i} \in \operatorname{End}_{\mathscr{H}_{r,p,n}}\left(V_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\right)$$

corresponds to the $\mathscr{H}_{r,n}$-homomorphism $V_{\mathbf{b}} \longrightarrow V_{\mathbf{b}} \otimes_{\mathscr{H}_{r,p,n}} \mathscr{H}_{r,n}$ given by

$$v_{\mathbf{b}}h \mapsto \sum_{s=0}^{p-1} \theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}}hT_0^{-s}) \otimes T_0^s,$$

for $h \in \mathscr{H}_{r,n}$. Using Proposition 3.11, and the explicit isomorphism given in Lemma 3.10,

$$\begin{aligned}\pi_j(\theta_{\mathbf{b}}^{-i})(v_{\mathbf{b}}) &= \sum_{s=0}^{p-1} \varepsilon^{jo_{\mathbf{b}}s}\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}}T_0^{-s})T_0^s = \sum_{s=0}^{p-1} \varepsilon^{jo_{\mathbf{b}}s}\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}})\varepsilon^{-iso_{\mathbf{b}}}T_0^{-s}T_0^s\\ &= \sum_{s=0}^{p-1} \varepsilon^{(j-i)so_{\mathbf{b}}}\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}}) = \delta_{ij}p\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}}).\end{aligned}$$

By assumption p does not divide the characteristic of K, so p is invertible in K. So we deduce that $\pi_i(\theta_{\mathbf{b}}^{-i})$ is actually an isomorphism from $V_{\mathbf{b}}$ onto $V_{\mathbf{b}}^{\sigma^{io_{\mathbf{b}}}}$. Essentially the same argument shows that if $x \in \mathscr{H}_{d,\mathbf{b}}$ then

$$\pi_j(x)(v_{\mathbf{b}}) = \delta_{j0}px \cdot v_{\mathbf{b}} = \delta_{j0}pv_{\mathbf{b}}\Theta_{\mathbf{b}}(x).$$

Therefore, $\pi_j(\theta_{\mathbf{b}}^{-i}x)(v_{\mathbf{b}}) = \delta_{ij}\delta_{j0}p^2\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}})\Theta_{\mathbf{b}}(x)$. Note that every homomorphism in $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}, V_{\mathbf{b}}^{\sigma^{io_{\mathbf{b}}}})$ can be decomposed into a composition of the isomorphism $\pi_i(\theta_{\mathbf{b}}^{-i})$ with an endomorphism in $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}) \cong \mathscr{H}_{d,\mathbf{b}}$. All of the claims in the theorem now follow. □

The algebra $\mathscr{E}_{d,\mathbf{b}}$ is generated by $\mathscr{H}_{d,\mathbf{b}}$ and $\theta_{\mathbf{b}}$ by Theorem 4.2. To make this more explicit, for $s = 1, 2, \dots, p$ let $T_i^{(s)}$ and $L_j^{(s)}$, for $1 \le i < b_s$ and $1 \le j \le b_s$, be the generators of $\mathscr{H}_{d,\mathbf{b}}$. That is,

$$T_i^{(s)} = 1^{\otimes s-1} \otimes T_i \otimes 1^{\otimes p-s} \quad \text{and} \quad L_j^{(s)} = 1^{\otimes s-1} \otimes L_j \otimes 1^{\otimes p-s},$$

interpreted as elements of $\mathscr{H}_{d,\mathbf{b}} = \mathscr{H}_{d,b_1}(\varepsilon\mathbf{Q}) \otimes \cdots \otimes \mathscr{H}_{d,b_p}(\varepsilon^p\mathbf{Q})$. The elements $T_i^{(s)}$ and $L_j^{(s)}$, for $1 \le s \le p$, $1 \le i < b_s$ and $1 \le j \le b_s$, generate $\mathscr{H}_{d,\mathbf{b}}$ subject to the relations implied by the defining relations for $\mathscr{H}_{r,n}$.

To determine the relations these elements satisfy in $\mathscr{E}_{d,\mathbf{b}}$ we need, at a minimum, to determine the commutation relations between elements and $\theta_{\mathbf{b}}$. Using Lemma 2.20, it is easy to deduce the following result.

Lemma 4.3. *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$, $1 \le s \le p$, $1 \le i < b_s$ *and* $1 \le j \le b_s$. *Then*

$$T_i^{(s)}\theta_{\mathbf{b}} = \begin{cases} \theta_{\mathbf{b}}T_i^{(s+o_{\mathbf{b}})}, & \text{if } s + o_{\mathbf{b}} \le p,\\ \theta_{\mathbf{b}}T_i^{(s+o_{\mathbf{b}}-p)}, & \text{if } s + o_{\mathbf{b}} > p,\end{cases}$$

$$L_j^{(s)}\theta_{\mathbf{b}} = \begin{cases} \varepsilon^{-o_{\mathbf{b}}}\theta_{\mathbf{b}}L_j^{(s+o_{\mathbf{b}})}, & \text{if } s + o_{\mathbf{b}} \le p,\\ \varepsilon^{-o_{\mathbf{b}}}\theta_{\mathbf{b}}L_j^{(s+o_{\mathbf{b}}-p)}, & \text{if } s + o_{\mathbf{b}} > p.\end{cases}$$

This lemma, when combined with the relation that $\theta_{\mathbf{b}}^{p_{\mathbf{b}}} = \varepsilon^{dn(p-\mathsf{o}_{\mathbf{b}})/2} z_{\mathbf{b}}$ is central in $\mathscr{E}_{d,\mathbf{b}}$ and the relations coming from $\mathscr{H}_{d,\mathbf{b}}$ gives a complete set of commutator relations for the generators of $\mathscr{E}_{d,\mathbf{b}}$. It would be interesting to know whether or not this gives a presentation for the algebra $\mathscr{E}_{d,\mathbf{b}}$.

Remark 4.4. Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $1 \le s, t \le p$ and $s \equiv t \pmod{\mathsf{o}_{\mathbf{b}}}$, so that $b_s = b_t$. Let π_{st} be the algebra isomorphism $\mathscr{H}_{d,b_s}^{(s)} \cong \mathscr{H}_{d,b_t}^{(t)}$ given by

$$T_i^{(s)} \mapsto T_i^{(t)} \quad \text{and} \quad T_0^{(s)} = L_1^{(s)} \mapsto \varepsilon^{s-t} T_0^{(t)}, \qquad \text{for } 1 \le i \le n-1.$$

Thus, π_{st} identifies the s^{th} tensor factor and the t^{th} tensor factor in $\mathscr{H}_{d,\mathbf{b}}$ and Lemma 4.3 says that conjugation by $\theta_{\mathbf{b}}$ coincides with the map π_{st}, where $t = s + \mathsf{o}_{\mathbf{b}}$ if $s + \mathsf{o}_{\mathbf{b}} \le p$; or $t = s + \mathsf{o}_{\mathbf{b}} - p$ if $s + \mathsf{o}_{\mathbf{b}} > p$.

Extend the equivalence relation $\sim_\sigma$ on $\mathscr{P}_{r,n}$ to $\mathscr{C}_{p,n}$ by defining $\mathbf{b} \sim_\sigma \mathbf{c}$ if $\mathbf{b} = \mathbf{c}\langle k\rangle$ for some $k \in \mathbb{Z}$, for $\mathbf{b}, \mathbf{c} \in \mathscr{C}_{p,n}$. Let $\mathscr{C}_{p,n}^\sigma = \mathscr{C}_{p,n}/\sim_\sigma$ be the set of $\sim_\sigma$-equivalence classes in $\mathscr{C}_{p,n}$. Once again, we write $\mathbf{b} \in \mathscr{C}_{p,n}^\sigma$ to indicate that $\mathbf{b}$ is a representative for an equivalence class in $\mathscr{C}_{p,n}^\sigma$.

Define $\mathscr{E}_d = \displaystyle\bigoplus_{\mathbf{b} \in \mathscr{C}_{p,n}^\sigma} \mathscr{E}_{d,\mathbf{b}}$. Note that $\mathscr{E}_d$ depends on the parameters q and $\mathbf{Q}^{\vee\varepsilon}$ and on n. Further, by definition, $\text{Mod-}\mathscr{E}_d = \bigoplus_{\mathbf{b} \in \mathscr{C}_{p,n}^\sigma} \text{Mod-}\mathscr{E}_{d,\mathbf{b}}$.

Corollary 4.5. *There is a Morita equivalence*

$$\mathtt{F}_{\mathscr{E}} : \text{Mod-}\mathscr{E}_d \longrightarrow \text{Mod-}\mathscr{H}_{r,p,n}; M \mapsto M \otimes_{\mathscr{E}_{d,\mathbf{b}}} V_{\mathbf{b}},$$

for $M \in \text{Mod-}\mathscr{E}_{d,\mathbf{b}}$, *and* $\mathbf{b} \in \mathscr{C}_{p,n}^\sigma$.

Proof. By Proposition 2.23(b), $\bigoplus_{\mathbf{b} \in \mathscr{C}_{p,n}} V_{\mathbf{b}}$ is a progenerator for $\mathscr{H}_{r,n}$. Moreover, if $\mathbf{b} \in \mathscr{C}_{p,n}$ then

$$V_{\mathbf{b}} \downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}} \cong V_{\mathbf{b}}^{\sigma^t} \downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}} \cong V_{\mathbf{b}\langle -t\rangle} \downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}},$$

for any $t \in \mathbb{Z}$ by Lemma 3.19. Therefore, $\bigoplus_{\mathbf{b} \in \mathscr{C}_{p,n}^\sigma} V_{\mathbf{b}}$ is a progenerator for $\mathscr{H}_{r,p,n}$ and, by well-known arguments, for example [5, §2.2], it induces the Morita equivalence $\mathtt{F}_{\mathscr{E}}$ above. □

We now describe the images of the Specht modules and simple modules of the algebra $\mathscr{H}_{r,p,n}$ under this Morita equivalence.

Let $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. By definition $\mathsf{o}_{\mathbf{b}} \mid \mathsf{o}_{\boldsymbol{\lambda}}$ and $\mathsf{o}_{\boldsymbol{\lambda}} \mid p$. Let $p_{\mathbf{b}/\boldsymbol{\lambda}} := p_{\mathbf{b}}/p_{\boldsymbol{\lambda}} = \mathsf{o}_{\boldsymbol{\lambda}}/\mathsf{o}_{\mathbf{b}} \in \mathbb{N}$. Then $p_{\mathbf{b}} = p_{\mathbf{b}/\boldsymbol{\lambda}} p_{\boldsymbol{\lambda}}$.

Definition 4.6. *Suppose that* $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, *for* $\mathbf{b} \in \mathscr{C}_{p,n}^\sigma$. *Define*

$$S^{\boldsymbol{\lambda}} = S_{\mathbf{b}}(\boldsymbol{\lambda}) \uparrow_{\mathscr{H}_{d,\mathbf{b}}}^{\mathscr{E}_{d,\mathbf{b}}} \quad \textit{and} \quad D^{\boldsymbol{\lambda}} = D_{\mathbf{b}}(\boldsymbol{\lambda}) \uparrow_{\mathscr{H}_{d,\mathbf{b}}}^{\mathscr{E}_{d,\mathbf{b}}}.$$

Define $\mathscr{E}_{d,\boldsymbol{\lambda}}$ *to be the subalgebra of* $\mathscr{E}_{d,\mathbf{b}}$ *generated by* $\mathscr{H}_{d,\mathbf{b}}$ *and* $(\theta_{\mathbf{b}})^{p_{\mathbf{b}/\boldsymbol{\lambda}}}$.

By definition $\mathscr{E}_{d,\boldsymbol{\lambda}} \cong \mathscr{E}_{d,\boldsymbol{\mu}}$ whenever $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$ and $p_{\boldsymbol{\lambda}} = p_{\boldsymbol{\mu}}$. Further, $\dim \mathscr{E}_{d,\boldsymbol{\lambda}} = p_{\boldsymbol{\lambda}} \dim \mathscr{H}_{d,\mathbf{b}}$ by Theorem 4.2. Notice that the maps $(\theta_{\mathbf{b}})^{p_{\mathbf{b}/\boldsymbol{\lambda}}}$ and $\theta_{\boldsymbol{\lambda}}$ agree when they are restricted to $S(\boldsymbol{\lambda})$.

Now fix generators $s_{\mathbf{b}}(\boldsymbol{\lambda})$ and $d_{\mathbf{b}}(\boldsymbol{\lambda})$ of $S_{\mathbf{b}}(\boldsymbol{\lambda})$ and $D_{\mathbf{b}}(\boldsymbol{\lambda})$, respectively, which we consider as elements of $\mathscr{E}_{d,\mathbf{b}}$. Motivated by Definition 3.47 and Theorem 3.48 define

$$S^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}} = s_{\mathbf{b}}(\boldsymbol{\lambda}) \prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\neq i}} \big((\theta_{\mathbf{b}})^{p_{\mathbf{b}/\boldsymbol{\lambda}}} - \mathfrak{g}_{\boldsymbol{\lambda}}\big)\mathscr{H}_{d,\mathbf{b}} \hookrightarrow \mathscr{E}_{d,\boldsymbol{\lambda}}$$

$$D^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}} = d_{\mathbf{b}}(\boldsymbol{\lambda}) \prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\neq i}} \big((\theta_{\mathbf{b}})^{p_{\mathbf{b}/\boldsymbol{\lambda}}} - \mathfrak{g}_{\boldsymbol{\lambda}}\big)\mathscr{H}_{d,\mathbf{b}} \hookrightarrow \mathscr{E}_{d,\boldsymbol{\lambda}}.$$

By Lemma 4.3, $S^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}$ and $D^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}$ are $\mathscr{E}_{d,\boldsymbol{\lambda}}$-submodules of $S^{\boldsymbol{\lambda}}$ and $D^{\boldsymbol{\lambda}}$, respectively. Moreover, it is easy to see that

$$S_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow^{\mathscr{E}_{d,\boldsymbol{\lambda}}}_{\mathscr{H}_{d,\mathbf{b}}} \cong \bigoplus_{i=1}^{p_{\boldsymbol{\lambda}}} S^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}} \quad\text{and}\quad D_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow^{\mathscr{E}_{d,\boldsymbol{\lambda}}}_{\mathscr{H}_{d,\mathbf{b}}} \cong \bigoplus_{i=1}^{p_{\boldsymbol{\lambda}}} D^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}.$$

Now define

$$S^{\boldsymbol{\lambda}}_{i,p} = S^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\uparrow^{\mathscr{E}_{d,\mathbf{b}}}_{\mathscr{E}_{d,\boldsymbol{\lambda}}} \quad\text{and}\quad D^{\boldsymbol{\lambda}}_{i,p} = D^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\uparrow^{\mathscr{E}_{d,\mathbf{b}}}_{\mathscr{E}_{d,\boldsymbol{\lambda}}}.$$

Let $\sim_{\mathbf{b}}$ be the equivalence relation on $\mathscr{P}_{d,\mathbf{b}}$ where if $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$ then $\boldsymbol{\mu}\sim_{\mathbf{b}}\boldsymbol{\lambda}$ if $\boldsymbol{\lambda}=\boldsymbol{\mu}\langle ko_{\mathbf{b}}\rangle$, for some $k\in\mathbb{Z}$. Let $\mathscr{P}^{\mathbf{b}}_{d,\mathbf{b}}$ be the set of $\sim_{\mathbf{b}}$-equivalence classes in $\mathscr{P}_{d,\mathbf{b}}$ and let $\mathscr{K}^{\mathbf{b}}_{d,\mathbf{b}}$ be the equivalence classes in $\mathscr{K}_{d,\mathbf{b}}$. Once again, we blur the distinction between equivalence classes in $\mathscr{P}^{\mathbf{b}}_{d,\mathbf{b}}$ and the multipartitions in these equivalence classes.

Lemma 4.7. *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, $\boldsymbol{\mu}\in\mathscr{K}^{\mathbf{b}}_{d,\mathbf{b}}$, $1\le i\le p_{\boldsymbol{\lambda}}$ and $1\le j\le p_{\boldsymbol{\mu}}$. Then $\mathtt{F}_{\mathscr{E}}S^{\boldsymbol{\lambda}}_{i,p}\cong S^{\boldsymbol{\lambda}}_i$ and $\mathtt{F}_{\mathscr{E}}D^{\boldsymbol{\mu}}_{j,p}\cong D^{\boldsymbol{\mu}}_j$. In particular,*

$$\{\, D^{\boldsymbol{\mu}}_{j,p} \mid \boldsymbol{\mu}\in\mathscr{K}^{\mathbf{b}}_{d,\mathbf{b}} \text{ and } 1\le j\le p_{\boldsymbol{\mu}} \,\}$$

is a complete set of pairwise non-isomorphic absolutely irreducible $\mathscr{E}_{d,\mathbf{b}}$-modules.

Proof. This follows directly from the definitions and standard properties of the Schur functor $\mathtt{F}_{\mathscr{E}}$. □

4.2. **A cyclotomic q-Schur algebra for $\mathscr{H}_{r,p,n}$.** The next step towards computing the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ is to lift our computations up to an analogue of the (cyclotomic) Schur algebra for $\mathscr{H}_{r,p,n}$. In this section we define such an algebra $\mathscr{S}_{r,p,n}$ and prove a basis theorem for it.

The cyclotomic Schur algebras [11] are defined as endomorphism algebras of certain permutation-like modules. We now define modules

$$M(\boldsymbol{\lambda}), M_{\mathbf{b}}(\boldsymbol{\lambda}) = M(\boldsymbol{\lambda}^{[1]})\otimes\cdots\otimes M(\boldsymbol{\lambda}^{[p]}) \text{ and } M^{\boldsymbol{\lambda}}_{\mathbf{b}} = \mathtt{H}_{\mathbf{b}}(M_{\mathbf{b}}(\boldsymbol{\lambda})) \cong v^{+}_{\mathbf{b}}M(\boldsymbol{\lambda}),$$

for $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ and $\mathbf{b}\in\mathscr{C}_{p,n}$. The *Schur algebras* for the three algebras $\mathscr{H}_{r,n}$, $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,p,n}$ are then defined to the endomorphism algebras of direct sums of these modules.

Definition 4.8. a) *The **cyclotomic q-Schur algebra** of $\mathscr{H}_{r,n}$ is the endomorphism algebra*

$$\mathscr{S}_{r,n} = \operatorname{End}_{\mathscr{H}_{r,n}}\Big(\bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{r,n}} M(\boldsymbol{\lambda})\Big).$$

b) *For* $\mathbf{b}\in\mathscr{C}_{p,n}$ *the* ***cyclotomic*** q***-Schur algebra*** *of* $\mathscr{H}_{d,\mathbf{b}}$ *is the endomorphism algebra*

$$\mathscr{S}_{d,\mathbf{b}}=\operatorname{End}_{\mathscr{H}_{d,\mathbf{b}}}\Big(\bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}}M_{\mathbf{b}}(\boldsymbol{\lambda})\Big).$$

c) *The* ***cyclotomic*** q***-Schur algebra*** *of* $\mathscr{H}_{r,p,n}$ *is the endomorphism algebra* $\mathscr{S}_{r,p,n}=\bigoplus_{\mathbf{b}\in\mathscr{C}_{p,n}}\mathscr{S}_{r,p,n}(\mathbf{b})$, *where*

$$\mathscr{S}_{r,p,n}(\mathbf{b})=\operatorname{End}_{\mathscr{H}_{r,p,n}}\Big(\bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}}M_{\mathbf{b}}^{\boldsymbol{\lambda}}\Big),$$

where $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ *is considered as an* $\mathscr{H}_{r,p,n}$*-module by restriction.*

The algebra $\mathscr{S}_{r,p,n}$ is new, generalizing the Schur algebras of type D introduced by the first author in [24]. The cyclotomic Schur algebra $\mathscr{S}_{r,n}=\mathscr{S}_{r,n}(\mathbf{Q}^{\vee\varepsilon})$ was introduced in [11]. By Definition 3.23, $M_{\mathbf{b}}(\boldsymbol{\lambda})=M(\boldsymbol{\lambda}^{[1]})\otimes\cdots\otimes M(\boldsymbol{\lambda}^{[p]})$ so that

$$\mathscr{S}_{d,\mathbf{b}}\cong\operatorname{End}_{\mathscr{H}_{d,\mathbf{b}}}\big(\bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}}M_{\mathbf{b}}(\boldsymbol{\lambda})\big)\cong\mathscr{S}_{d,\mathbf{b}_1}(\varepsilon\mathbf{Q})\otimes\cdots\otimes\mathscr{S}_{d,b_p}(\varepsilon^p\mathbf{Q}).$$

Moreover, applying the functor $\mathtt{H}_{\mathbf{b}}$ shows that

$$\mathscr{S}_{d,\mathbf{b}}\cong\operatorname{End}_{\mathscr{H}_{r,n}}\Big(\bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}}M_{\mathbf{b}}^{\boldsymbol{\lambda}}\Big).\tag{4.9}$$

Hence, we can — and do! — consider $\mathscr{S}_{d,\mathbf{b}}$ as a subalgebra of $\mathscr{S}_{r,p,n}$.

Recall that after Definition 4.1 we defined $\theta_{\mathbf{b}}=\theta_{0,o_{\mathbf{b}}}\in\operatorname{End}_{\mathscr{H}_{r,p_{\mathbf{b}},n}}(V_{\mathbf{b}})$. By definition, $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ is a submodule of $V_{\mathbf{b}}$. We next show that $\theta_{\mathbf{b}}$ maps $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ to $M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$.

Lemma 4.10. *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and* $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$. *Then* $\theta_{\mathbf{b}}$ *restricts to give an* $\mathscr{H}_{r,p,n}$*-homomorphism from* $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ *to* $M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$.

Proof. Let $Y_{\mathbf{b}}=Y_{0,o_{\mathbf{b}}}=Y_{o_{\mathbf{b}}}\dots Y_1$. Then $\theta_{\mathbf{b}}(v)=\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}v)$, for all $v\in V_{\mathbf{b}}$. By definition, $M_{\mathbf{b}}^{\boldsymbol{\lambda}}=v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}}\mathscr{H}_{r,n}$ and $v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}}=v_{\mathbf{b}}\Theta_{\mathbf{b}}(u_{\boldsymbol{\lambda},\mathbf{b}}^{+}x_{\boldsymbol{\lambda},\mathbf{b}})=\widehat{\Theta}_{\mathbf{b}}(x_{\boldsymbol{\lambda},\mathbf{b}}u_{\boldsymbol{\lambda},\mathbf{b}}^{+})v_{\mathbf{b}}$, where these elements are defined just before Definition 3.23. Therefore, it is enough to prove that $\theta_{\mathbf{b}}(v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}})=\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}})$ belongs to $M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$. Using (2.24) we compute

$$\begin{aligned}
Y_{\mathbf{b}}v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}}&=Y_{\mathbf{b}}v_{\mathbf{b}}\Theta_{\mathbf{b}}(u_{\boldsymbol{\lambda},\mathbf{b}}^{+}x_{\boldsymbol{\lambda},\mathbf{b}})\\
&=\widehat{\Theta}_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}(u^{+}_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle}x_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle})Y_{\mathbf{b}}v_{\mathbf{b}}, &&\text{by Lemma 2.20,}\\
&=\widehat{\Theta}_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}(u^{+}_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle}x_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle})v^{(o_{\mathbf{b}})}_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}Y_{\mathbf{b}}^{*}, &&\text{by Corollary 2.15.}
\end{aligned}$$

Hence, using Lemma 3.18 there exists an integer $c\in\mathbb{Z}$ such that

$$\begin{aligned}
\theta_{\mathbf{b}}(v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}})&=\varepsilon^{c}v_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}\Theta_{\mathbf{b}}(u^{+}_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle}x_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle})\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}^{*})\\
&\in v^{+}_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}u^{+}_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}^{*}).
\end{aligned}$$

Thus, $\theta_{\mathbf{b}}(v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}})\in M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$. Moreover, this map is surjective because left multiplication by $Y_{\mathbf{b}}$, and hence by $\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}^{*})$, is invertible by Lemma 2.28 and Lemma 2.25. As $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ and $M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$ are both free and of the same rank the proof is complete. $\square$

Recall from Lemma 2.25 that $z_{\mathbf{b}}$ is a central element of $\mathscr{H}_{d,\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}_{p,n}$. Consequently, if $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ then

$$z_{\mathbf{b}} \cdot v_{\mathbf{b}}^+ u_{\boldsymbol{\lambda}}^+ x_{\boldsymbol{\lambda}} = (z_{\mathbf{b}} u_{\boldsymbol{\lambda},\mathbf{b}}^+ x_{\boldsymbol{\lambda},\mathbf{b}}) \cdot v_{\mathbf{b}} = (u_{\boldsymbol{\lambda},\mathbf{b}}^+ x_{\boldsymbol{\lambda},\mathbf{b}} z_{\mathbf{b}}) \cdot v_{\mathbf{b}} = (u_{\boldsymbol{\lambda},\mathbf{b}}^+ x_{\boldsymbol{\lambda},\mathbf{b}}) \cdot v_{\mathbf{b}} \Theta_{\mathbf{b}}(z_{\mathbf{b}}) \in M_{\mathbf{b}}^{\boldsymbol{\lambda}}.$$

Therefore, left multiplication by $z_{\mathbf{b}}$ induces a homomorphism in $\operatorname{End}_{\mathscr{H}_{r,n}}(M_{\mathbf{b}}^{\boldsymbol{\lambda}})$.

Definition 4.11. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$. Define maps $\vartheta_{\mathbf{b}}$ and $\zeta_{\mathbf{b}}$ in $\mathscr{S}_{r,p,n}(\mathbf{b})$ by*

$$\vartheta_{\mathbf{b}}(m) = \theta_{\mathbf{b}}(m) \quad \textit{and} \quad \zeta_{\mathbf{b}}(m) = z_{\mathbf{b}} \cdot m,$$

for $m \in M_{\mathbf{b}}^{\boldsymbol{\lambda}}$, and $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$.

Using this definition and Lemma 3.44 we obtain:

Lemma 4.12. *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$. Then $\zeta_{\mathbf{b}}$ is central in $\mathscr{S}_{r,p,n}$ and*

$$\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}} = \varepsilon^{\frac{1}{2} d o_{\mathbf{b}} n (p_{\mathbf{b}}-1)} \zeta_{\mathbf{b}}.$$

As remarked in (4.9) above, $\mathscr{S}_{d,\mathbf{b}} \cong \operatorname{End}_{\mathscr{H}_{r,n}}\big(\bigoplus_{\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}} M_{\mathbf{b}}^{\boldsymbol{\lambda}}\big)$ and we view $\mathscr{S}_{d,\mathbf{b}}$ as a subalgebra of $\mathscr{S}_{r,p,n}(\mathbf{b})$ via this isomorphism.

Theorem 4.13. *As a K-algebra, $\mathscr{S}_{r,p,n}(\mathbf{b})$ is generated by $\mathscr{S}_{d,\mathbf{b}}$ and the endomorphism $\vartheta_{\mathbf{b}}$. Moreover, if $\{\, x_i \mid i \in I \,\}$ is a K-basis of $\mathscr{S}_{d,\mathbf{b}}$ then*

$$\{\, x_i \vartheta_{\mathbf{b}}^k \mid i \in I \textit{ and } 0 \le k < p_{\mathbf{b}} \,\}$$

is a K-basis of $\mathscr{S}_{r,p,n}(\mathbf{b})$. In particular, $\dim \mathscr{S}_{r,p,n}(\mathbf{b}) = p_{\mathbf{b}} \dim \mathscr{S}_{d,\mathbf{b}}$.

Proof. This can be proved by repeating the argument of Theorem 4.2. □

4.3. **Weyl modules, simple modules and Schur functors.** This section lifts the problem of computing the p-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ up to $\mathscr{S}_{r,p,n}$ by constructing Weyl modules, simple modules for $\mathscr{S}_{r,p,n}$. We then construct an analogue of the Schur functor to relate the categories of $\mathscr{S}_{r,p,n}$-modules and $\mathscr{H}_{r,p,n}$-modules, via the category of $\mathscr{E}_d$-modules.

The cyclotomic Schur algebra $\mathscr{S}_{r,n}$ is a quasi-hereditary cellular algebra with basis $\{\, \varphi_{\mathsf{ST}} \mid \mathsf{S} \in \mathcal{T}_0(\boldsymbol{\lambda}, \boldsymbol{\mu}), \mathsf{T} \in \mathcal{T}_0(\boldsymbol{\lambda}, \boldsymbol{\nu}) \text{ for } \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}_{r,n} \,\}$, where $\mathcal{T}_0(\boldsymbol{\lambda}, \boldsymbol{\tau})$ is the set of *semistandard $\boldsymbol{\lambda}$-tableaux of type $\boldsymbol{\tau}$* for $\boldsymbol{\tau} \in \mathscr{P}_{r,n}$; see [11, Definition 4.4 and Theorem 6.6]. In this paper we do not need the precise combinatorial definition of semistandard tableaux. For our purposes it is enough to know that if $x = u_{\boldsymbol{\tau}}^+ x_{\boldsymbol{\tau}} h \in M(\boldsymbol{\tau})$, and $\mathsf{S} \in \mathcal{T}_0(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $\mathsf{T} \in \mathcal{T}_0(\boldsymbol{\lambda}, \boldsymbol{\nu})$, then

$$\varphi_{\mathsf{ST}}(x) = \delta_{\boldsymbol{\nu\tau}} m_{\mathsf{ST}} h,$$

where m_{ST} is a certain element of $M(\boldsymbol{\mu})$.

For each $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ there is a **Weyl module** $\Delta(\boldsymbol{\lambda})$, which is a cell module for $\mathscr{S}_{r,n}$. Let $L(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda}) / \operatorname{rad} \Delta(\boldsymbol{\lambda})$, where $\operatorname{rad} \Delta(\boldsymbol{\lambda})$ is the Jacobson radical of $\Delta(\boldsymbol{\lambda})$. Then $\{\, L(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{S}_{r,n}$-modules. Further, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{r,n}$ then $L(\boldsymbol{\mu})$ is the simple head of $\Delta(\boldsymbol{\mu})$ and

$$[\Delta(\boldsymbol{\lambda}) : L(\boldsymbol{\mu})] = \begin{cases} 1, & \text{if } \boldsymbol{\lambda} = \boldsymbol{\mu}, \\ 0, & \text{if } \boldsymbol{\lambda} \not\trianglerighteq \boldsymbol{\mu}. \end{cases} \tag{4.14}$$

All of these facts are proved in [11, §6].

Similarly, for $\mathbf{b}\in\mathscr{C}_{p,n}$ let $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ and $L_{\mathbf{b}}(\boldsymbol{\lambda})$ be the Weyl modules and the irreducible modules of $\mathscr{S}_{d,\mathbf{b}}$, for $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$. For $1\le t\le p$, $\lambda,\nu,\mu\in\mathscr{P}_{d,b_t}$ and $\mathsf{S}\in\mathcal{T}_0(\lambda,\mu)$, $\mathsf{T}\in\mathcal{T}_0(\lambda,\nu)$, let $\varphi^{(t)}_{\mathsf{ST}}$ be the corresponding element of $\mathscr{S}_{d,\mathbf{b}}$ given by

$$\varphi^{(t)}_{\mathsf{ST}}(x_1\otimes\cdots\otimes x_p)=x_1\otimes\cdots\otimes x_{t-1}\otimes\varphi_{\mathsf{ST}}(x_t)\otimes x_{t+1}\otimes\cdots\otimes x_p.$$

Lemma 4.15. *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$, $1\le s\le p$ and that $\mathsf{S}\in\mathcal{T}_0(\lambda,\mu)$, and $\mathsf{T}\in\mathcal{T}_0(\lambda,\nu)$, where $\lambda,\mu,\nu\in\mathscr{P}_{d,b_s}$. Then*

$$\varphi^{(s)}_{\mathsf{ST}}\vartheta_{\mathbf{b}}=\begin{cases}\varepsilon^{-\mathsf{o}_{\mathbf{b}}k_{\lambda,\nu}}\vartheta_{\mathbf{b}}\varphi^{(s+\mathsf{o}_{\mathbf{b}})}_{\mathsf{ST}}, & \text{if } s+\mathsf{o}_{\mathbf{b}}\le p,\\ \varepsilon^{-\mathsf{o}_{\mathbf{b}}k_{\lambda,\nu}}\vartheta_{\mathbf{b}}\varphi^{(s+\mathsf{o}_{\mathbf{b}}-p)}_{\mathsf{ST}}, & \text{if } s+\mathsf{o}_{\mathbf{b}}>p.\end{cases}$$

where $k_{\lambda,\nu}=\sum_{s=1}^{d-1}\sum_{t=1}^{s}(|\lambda^{(t)}|-|\nu^{(t)}|)$.

Proof. We first note that $\mathbf{b}\langle\mathsf{o}_{\mathbf{b}}\rangle=\mathbf{b}$, so that the notations $\varphi^{(s+\mathsf{o}_{\mathbf{b}})}_{\mathsf{ST}}$ and $\varphi^{(s+\mathsf{o}_{\mathbf{b}}-p)}_{\mathsf{ST}}$ make sense. As the map φ_{ST} is given by left multiplication by an element of $\mathscr{H}_{d,\mathbf{b}}$, the result follows from Lemma 4.3. (In the sequel we only need to know that the scalar $\varepsilon^{-\mathsf{o}_{\mathbf{b}}k_{\lambda,\nu}}$ above is equal to $\varepsilon^{\mathsf{o}_b k}$, for some $k\in\mathbb{Z}$, which is a consequence of Lemma 4.3. That $k=k_{\lambda,\nu}$ can be determined using the definition of m_{ST} from [11].) □

Remark 4.16. Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$ and $1\le s,t\le p$ and $s\equiv t\pmod{\mathsf{o}_{\mathbf{b}}}$, so that $b_s=b_t$. Just as in Remark 4.4, if we let π'_{st} be the algebra isomorphism ${\mathscr{S}_{d,b_s}}^{(s)}\cong{\mathscr{S}_{d,b_t}}^{(t)}$ given by $\varphi^{(s)}_{\mathsf{ST}}\mapsto\varepsilon^{-\mathsf{o}_{\mathbf{b}}k_{\lambda,\nu}}\varphi^{(t)}_{\mathsf{ST}}$, for S and T as above. Then $\vartheta_{\mathbf{b}}$ coincides with π'_{st}, where $t=s+\mathsf{o}_{\mathbf{b}}$ if $s+\mathsf{o}_{\mathbf{b}}\le p$; or $t=s+\mathsf{o}_{\mathbf{b}}-p$ if $s+\mathsf{o}_{\mathbf{b}}>p$.

For each multipartition $\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$ the identity map $\varphi_{\boldsymbol{\mu}}\colon M_{\mathbf{b}}(\boldsymbol{\mu})\longrightarrow M_{\mathbf{b}}(\boldsymbol{\mu})$ belongs to $\mathscr{S}_{d,\mathbf{b}}$. Then φ_μ is an idempotent in $\mathscr{S}_{d,\mathbf{b}}$ and $\sum_{\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}}\varphi_{\boldsymbol{\mu}}$ is the identity element of $\mathscr{S}_{d,\mathbf{b}}$. If M is a $\mathscr{S}_{d,\mathbf{b}}$-module then M has a **weight space** decomposition

$$M=\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}}M_{\boldsymbol{\mu}},\qquad\text{where } M_{\boldsymbol{\mu}}=M\varphi_{\boldsymbol{\mu}}.$$

Recall from (2.27) that $\boldsymbol{\omega}_{\mathbf{b}}=({\boldsymbol{\omega}_{\mathbf{b}}}^{[1]},\cdots,{\boldsymbol{\omega}_{\mathbf{b}}}^{[p]})$ is the unique multipartition in $\mathscr{P}_{d,\mathbf{b}}$ such that $\boldsymbol{\mu}\trianglerighteq\boldsymbol{\omega}_{\mathbf{b}}$ for all $\mu\in\mathscr{P}_{d,\mathbf{b}}$. By definition, $\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$ is the identity map on $\mathscr{H}_{d,\mathbf{b}}$ so that $\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}\mathscr{S}_{d,\mathbf{b}}\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}\cong\mathscr{H}_{d,\mathbf{b}}$. Hence, we have a **Schur functor**

(4.17) $$\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}\colon\text{Mod-}\mathscr{S}_{d,\mathbf{b}}\longrightarrow\text{Mod-}\mathscr{H}_{d,\mathbf{b}};M\mapsto M_{\boldsymbol{\omega}_{\mathbf{b}}},\qquad\text{for } M\in\text{Mod-}\mathscr{S}_{d,\mathbf{b}}.$$

By [11, Corollary 6.14], the Weyl module $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ has a basis

$$\{\,\varphi_{\mathsf{S}}\mid\mathsf{S}\in\mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\mu})\text{ for }\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}\,\}$$

such that $\{\,\varphi_{\mathsf{S}}\mid\mathsf{S}\in\mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\mu})\,\}$ is a basis for the $\boldsymbol{\mu}$-weight space of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$. This implies that $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}(\Delta_{\mathbf{b}}(\boldsymbol{\lambda}))\cong S_{\mathbf{b}}(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$; see [26, Proposition 2.17]. Hence, $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}(L_{\mathbf{b}}(\boldsymbol{\lambda}))\cong D_{\mathbf{b}}(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda}\in\mathscr{K}_{d,\mathbf{b}}$, since $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}$ is exact.

There is a unique semistandard $\boldsymbol{\lambda}$-tableau $\mathsf{T}^{\boldsymbol{\lambda}}$ of type $\boldsymbol{\lambda}$ and $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ is a "highest weight vector" in $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$. In particular, $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ generates $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$.

Lemma 4.18. *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$. Then*

$$\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\zeta_{\mathbf{b}}=\mathfrak{f}_{\boldsymbol{\lambda}}\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\quad\textit{and}\quad\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}}=(\mathfrak{g}_{\boldsymbol{\lambda}})^{p_{\mathbf{b}}}\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}.$$

Proof. By [26, (2.18)], the Weyl module $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ can be identified with a set of maps from $\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}}M_{\mathbf{b}}(\boldsymbol{\mu})$ to $S_{\mathbf{b}}(\boldsymbol{\lambda})$ in such a way that $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ is identified with the natural projection map $M_{\mathbf{b}}(\boldsymbol{\lambda})\longrightarrow S_{\mathbf{b}}(\boldsymbol{\lambda})$. Hence, $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\zeta_{\mathbf{b}}=\mathfrak{f}_{\boldsymbol{\lambda}}\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ by Proposition 3.4 and $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}}=(\mathfrak{g}_{\boldsymbol{\lambda}})^{p_{\mathbf{b}}}\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ by Corollary 3.46 □

By Theorem 4.13, the subspaces $\{\mathscr{S}_{d,\mathbf{b}}, \vartheta_{\mathbf{b}}\mathscr{S}_{d,\mathbf{b}}, \dots, (\vartheta_{\mathbf{b}})^{p_{\mathbf{b}}-1}\mathscr{S}_{d,\mathbf{b}}\}$ define a $\mathbb{Z}/p_{\mathbf{b}}\mathbb{Z}$-graded Clifford system for $\mathscr{S}_{r,p,n}(\mathbf{b})$. In particular, conjugation with $\vartheta_{\mathbf{b}}$ defines an algebra automorphism of $\mathscr{S}_{d,\mathbf{b}}$. For any $\mathscr{S}_{d,\mathbf{b}}$-module M let $M^{\vartheta_{\mathbf{b}}}$ be the $\mathscr{S}_{d,\mathbf{b}}$-module obtained by twisting the action of $\mathscr{S}_{d,\mathbf{b}}$ by $\vartheta_{\mathbf{b}}$.

Lemma 4.19. *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}_{p,n}$. Then*

$$\Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}} \cong \Delta_{\mathbf{b}}(\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle) \quad \textit{and} \quad L_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}} \cong L_{\mathbf{b}}(\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle)$$

as $\mathscr{S}_{d,\mathbf{b}}$-modules.

Proof. This follows directly from Lemma 4.15 and Remark 4.16. □

The following definitions mirror the constructions of the Specht modules and irreducible $\mathscr{E}_{d,\mathbf{b}}$-modules given by Definition 4.6.

Definition 4.20. *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}_{p,n}$. Define*

$$\Delta^{\boldsymbol{\lambda}} = \Delta_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow_{\mathscr{S}_{d,\mathbf{b}}}^{\mathscr{S}_{r,p,n}(\mathbf{b})} \quad \textit{and} \quad L^{\boldsymbol{\lambda}} = L_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow_{\mathscr{S}_{d,\mathbf{b}}}^{\mathscr{S}_{r,p,n}(\mathbf{b})}.$$

Let $\hat{\sigma}$ be the automorphism of $\mathscr{S}_{r,p,n}(\mathbf{b})$ which, using Theorem 4.13, is defined on generators by

$$(x\vartheta_{\mathbf{b}}^k)^{\hat{\sigma}}x = \varepsilon^{ko_{\mathbf{b}}}x\vartheta_{\mathbf{b}}^k, \qquad \text{for all } x \in \mathscr{S}_{d,\mathbf{b}} \text{ and } 0 \le k < p_{\mathbf{b}}.$$

By definition, $\hat{\sigma}$ restricts to the identity map on $\mathscr{S}_{d,\mathbf{b}}$. By Lemma 3.10 there is an isomorphism of $\mathscr{S}_{r,p,n}(\mathbf{b})$-$\mathscr{S}_{r,p,n}(\mathbf{b})$-bimodules,

$$\mathscr{S}_{r,p,n}(\mathbf{b}) \otimes_{\mathscr{S}_{d,\mathbf{b}}} \mathscr{S}_{r,p,n}(\mathbf{b}) \cong \bigoplus_{j=1}^{p_{\mathbf{b}}} \big(\mathscr{S}_{r,p,n}(\mathbf{b})\big)^{\hat{\sigma}^j}, \tag{4.21}$$

such that the left $\mathscr{S}_{r,p,n}(\mathbf{b})$-module structure on $\big(\mathscr{S}_{r,p,n}(\mathbf{b})\big)^{\hat{\sigma}^j}$ is given by left multiplication and the right action is twisted by $\hat{\sigma}^j$.

Recall that if $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ then $p_{\mathbf{b}/\boldsymbol{\lambda}} = p_{\mathbf{b}}/p_{\boldsymbol{\lambda}}$. Let $\mathscr{S}_{d,\boldsymbol{\lambda}}$ be the subalgebra of $\mathscr{S}_{r,p,n}$ generated by $\mathscr{S}_{d,\mathbf{b}}$ and $\vartheta_{\boldsymbol{\lambda}} = \vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol{\lambda}}}$. Let $\overline{\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}}$ be the image of $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ in $L_{\mathbf{b}}(\boldsymbol{\lambda})$. For $1 \le i \le p_{\boldsymbol{\lambda}}$ define

$$\Delta_{i,p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}} = \varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} \prod_{\substack{1 \le t \le p_{\boldsymbol{\lambda}} \\ t \ne i}} \Big(\vartheta_{\boldsymbol{\lambda}} - \mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{o_{\boldsymbol{\lambda}}t}\Big)\mathscr{S}_{d,\mathbf{b}} \hookrightarrow \mathscr{S}_{d,\boldsymbol{\lambda}},$$

$$L_{i,p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}} = \overline{\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}} \prod_{\substack{1 \le t \le p_{\boldsymbol{\lambda}} \\ t \ne i}} \Big(\vartheta_{\boldsymbol{\lambda}} - \mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{o_{\boldsymbol{\lambda}}t}\Big)\mathscr{S}_{d,\mathbf{b}} \hookrightarrow \mathscr{S}_{d,\boldsymbol{\lambda}}.$$

Then, by Lemma 4.15 and Lemma 4.18, $\Delta_{i,p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}}$ and $L_{i,p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}}$ are $\mathscr{S}_{d,\boldsymbol{\lambda}}$-submodules of $\Delta^{\boldsymbol{\lambda}}$ and $L^{\boldsymbol{\lambda}}$, respectively. Next, for $1 \le i \le p_{\boldsymbol{\lambda}}$ define

$$\Delta_{i,p}^{\boldsymbol{\lambda}} = \Delta_{i,p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}}\uparrow_{\mathscr{S}_{d,\boldsymbol{\lambda}}}^{\mathscr{S}_{r,p,n}(\mathbf{b})} \quad \text{and} \quad L_{i,p}^{\boldsymbol{\lambda}} = L_{i,p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}}\uparrow_{\mathscr{S}_{d,\boldsymbol{\lambda}}}^{\mathscr{S}_{r,p,n}(\mathbf{b})}.$$

As we will see (cf. Lemma 3.11), each $L_{i,p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}}$ is an irreducible $\mathscr{S}_{d,\boldsymbol{\lambda}}$-module and each $L_{i,p}^{\boldsymbol{\lambda}}$ is a irreducible $\mathscr{S}_{r,p,n}$-module.

Proposition 4.22. *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}_{p,n}$, and let $\hat{\sigma}_{\boldsymbol{\lambda}} = (\hat{\sigma})^{p_{\mathbf{b}/\boldsymbol{\lambda}}}$. Then:*

a) *if* $1 \le i \le p_{\boldsymbol{\lambda}}$ *then*

$$\big(\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\big)^{\hat\sigma_{\boldsymbol{\lambda}}} \cong \Delta^{\boldsymbol{\lambda}}_{i+1,p_{\boldsymbol{\lambda}}}, \qquad \big(\Delta^{\boldsymbol{\lambda}}_{i,p}\big)^{\hat\sigma_{\boldsymbol{\lambda}}} \cong \Delta^{\boldsymbol{\lambda}}_{i+1,p},$$
$$\big(L^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\big)^{\hat\sigma_{\boldsymbol{\lambda}}} \cong L^{\boldsymbol{\lambda}}_{i+1,p_{\boldsymbol{\lambda}}} \qquad \big(L^{\boldsymbol{\lambda}}_{i,p}\big)^{\hat\sigma_{\boldsymbol{\lambda}}} \cong L^{\boldsymbol{\lambda}}_{i+1,p}.$$

b) $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow^{\mathscr{S}_{d,\boldsymbol{\lambda}}}_{\mathscr{S}_{d,\mathbf{b}}} \cong \oplus_{i=1}^{p_{\boldsymbol{\lambda}}}\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}$ *and* $L_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow^{\mathscr{S}_{d,\boldsymbol{\lambda}}}_{\mathscr{S}_{d,\mathbf{b}}} \cong \oplus_{i=1}^{p_{\boldsymbol{\lambda}}}L^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}$. *Moreover, there is a unique* $\mathscr{S}_{d,\mathbf{b}}$*-module isomorphism* $\Delta_{\mathbf{b}}(\boldsymbol{\lambda}) \longrightarrow \Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\downarrow^{\mathscr{S}_{d,\boldsymbol{\lambda}}}_{\mathscr{S}_{d,\mathbf{b}}}$ *such that*

$$\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} \mapsto \varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} \prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}} \Big(\vartheta_{\boldsymbol{\lambda}} - \mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{\mathsf{o}_{\boldsymbol{\lambda}}t}\Big).$$

This latter map induces an isomorphism $L_{\mathbf{b}}(\boldsymbol{\lambda}) \longrightarrow L^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\downarrow^{\mathscr{S}_{d,\boldsymbol{\lambda}}}_{\mathscr{S}_{d,\mathbf{b}}}$.

c) $\Delta^{\boldsymbol{\lambda}} = \Delta^{\boldsymbol{\lambda}}_{1,p} \oplus \cdots \oplus \Delta^{\boldsymbol{\lambda}}_{p_{\boldsymbol{\lambda}},p}$ *and* $L^{\boldsymbol{\lambda}} = L^{\boldsymbol{\lambda}}_{1,p} \oplus \cdots \oplus L^{\boldsymbol{\lambda}}_{p_{\boldsymbol{\lambda}},p}$ *as* $\mathscr{S}_{r,p,n}$*-modules.*

d) $\Delta^{\boldsymbol{\lambda}} \cong \Delta^{\boldsymbol{\lambda}\langle \mathsf{o}_{\mathbf{b}}\rangle}$ *and* $L^{\boldsymbol{\lambda}} \cong L^{\boldsymbol{\lambda}\langle \mathsf{o}_{\mathbf{b}}\rangle}$ *as* $\mathscr{S}_{r,p,n}$*-modules.*

Proof. We only prove the results for the Weyl modules. The other cases can be proved using similar arguments or using the fact that twisting by $\hat\sigma$ is an exact functor.

By Lemma 4.19, we know that $(\Delta_{\mathbf{b}}(\boldsymbol{\lambda}))^{\vartheta_{\boldsymbol{\lambda}}} \cong \Delta_{\mathbf{b}}(\boldsymbol{\lambda}\langle \mathsf{o}_{\boldsymbol{\lambda}}\rangle) = \Delta_{\mathbf{b}}(\boldsymbol{\lambda})$. Therefore,

$$\begin{aligned}\Delta^{\boldsymbol{\lambda}\langle \mathsf{o}_{\mathbf{b}}\rangle} &= \Delta_{\mathbf{b}}(\boldsymbol{\lambda}\langle \mathsf{o}_{\mathbf{b}}\rangle)\uparrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} \cong \Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}}\uparrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}\\ &\cong \big(\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}\big)^{\vartheta_{\mathbf{b}}} = \big(\Delta^{\boldsymbol{\lambda}}\big)^{\vartheta_{\mathbf{b}}} \cong \Delta^{\boldsymbol{\lambda}}.\end{aligned}$$

This proves (d).

Arguing as in Theorem 3.48, it is easy to see that $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} \in \Delta^{\boldsymbol{\lambda}}_{1,p} + \cdots + \Delta^{\boldsymbol{\lambda}}_{p_{\boldsymbol{\lambda}},p}$. Hence, $\Delta^{\boldsymbol{\lambda}} = \Delta^{\boldsymbol{\lambda}}_{1,p} + \cdots + \Delta^{\boldsymbol{\lambda}}_{p_{\boldsymbol{\lambda}},p}$. On the other hand, if $1 \le i \le p_{\boldsymbol{\lambda}}$ and $f \in \mathscr{S}_{d,\mathbf{b}}$ then the isomorphisms in Remark 4.16, together with the fact that $\boldsymbol{\lambda}\langle \mathsf{o}_{\boldsymbol{\lambda}}\rangle = \boldsymbol{\lambda}$, imply that $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} f = 0$ if and only if $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}(\vartheta^i_{\boldsymbol{\lambda}} f \vartheta^{-i}_{\boldsymbol{\lambda}}) = 0$. It follows that the map

$$\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} \mapsto \varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\Bigg(\prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}} \Big(\vartheta_{\boldsymbol{\lambda}} - \mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{\mathsf{o}_{\boldsymbol{\lambda}}t}\Big)\Bigg)$$

extends uniquely to a $\mathscr{S}_{d,\mathbf{b}}$-module surjection $\rho_i : \Delta_{\mathbf{b}}(\boldsymbol{\lambda}) \twoheadrightarrow \Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\downarrow^{\mathscr{S}_{d,\boldsymbol{\lambda}}}_{\mathscr{S}_{d,\mathbf{b}}}$, for $1 \le i \le p_{\boldsymbol{\lambda}}$. In particular, $\dim \Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}} \le \dim \Delta_{\mathbf{b}}(\boldsymbol{\lambda})$. By construction, however, $\dim \Delta^{\boldsymbol{\lambda}} = p_{\boldsymbol{\lambda}} \dim \Delta_{\mathbf{b}}(\boldsymbol{\lambda})$. Therefore, the maps ρ_i, for $1 \le i \le p_{\boldsymbol{\lambda}}$, are all isomorphisms. This proves (b), while (c) follows easily from the definitions and (b).

It remains to prove part (a). Suppose that $1 \le i \le p_{\boldsymbol{\lambda}}$. The definition of $\hat\sigma$ implies that if $f \in \mathscr{S}_{r,p,n}(\mathbf{b})$ then $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} f = 0$ if and only if $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} f^{\hat\sigma_{\boldsymbol{\lambda}}} = 0$. Therefore, the map

$$\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} \prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i+1}} \Big(\vartheta_{\boldsymbol{\lambda}} - \mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{\mathsf{o}_{\boldsymbol{\lambda}}t}\Big) f \mapsto \varphi_{\mathsf{T}^{\boldsymbol{\lambda}}} \prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}} \Big(\vartheta_{\boldsymbol{\lambda}} - \mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{\mathsf{o}_{\boldsymbol{\lambda}}t}\Big) f^{\hat\sigma_{\boldsymbol{\lambda}}}$$

is a well-defined $\mathscr{S}_{r,p,n}(\mathbf{b})$-module homomorphism from $\Delta^{\boldsymbol{\lambda}}_{i+1,p}$ onto $\big(\Delta^{\boldsymbol{\lambda}}_{i,p}\big)^{\hat\sigma_{\boldsymbol{\lambda}}}$. Similarly, one can prove that $\big(\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\big)^{\hat\sigma_{\boldsymbol{\lambda}}} \cong \Delta^{\boldsymbol{\lambda}}_{i+1,p_{\boldsymbol{\lambda}}}$. □

The proof of Proposition 4.22(a) yields the following.

Corollary 4.23. *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ and that $1 \le i \le p_{\boldsymbol{\lambda}}$. Then, as a K-vector space*

$$\Delta^{\boldsymbol{\lambda}}_{i,p} \cong \Delta_{\mathbf{b}}(\boldsymbol{\lambda}) \oplus \Delta_{\mathbf{b}}(\boldsymbol{\lambda})\vartheta_{\mathbf{b}} \oplus \cdots \oplus \Delta_{\mathbf{b}}(\boldsymbol{\lambda})\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}/\boldsymbol{\lambda}-1},$$

Moreover, the action of $\mathscr{S}_{r,p,n}(\mathbf{b})$ on $\Delta^{\boldsymbol{\lambda}}_{i,p}$ is uniquely determined by

a) $\Delta^{\boldsymbol{\lambda}}_{i,p}\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} \cong \Delta_{\mathbf{b}}(\boldsymbol{\lambda}) \oplus \Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{-1}} \oplus \cdots \oplus \Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{1-p_{\mathbf{b}}/\boldsymbol{\lambda}}}$*;*

b) $(x\vartheta_{\mathbf{b}}^{j})\vartheta_{\mathbf{b}}^{t} = x\vartheta_{\mathbf{b}}^{j+t}$*, for all $x \in \Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ and $j,t \in \mathbb{Z}$;*

c) *$\vartheta_{\boldsymbol{\lambda}}$ acts as the scalar $\mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{io_{\boldsymbol{\lambda}}}$ on the highest weight vector of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda}) \hookrightarrow \Delta^{\boldsymbol{\lambda}}_{i,p}$.*

Analogous statements hold for the simple module $L^{\boldsymbol{\lambda}}_{i,p}$.

Proof. By definition,

$$\Delta^{\boldsymbol{\lambda}}_{i,p} \cong \Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}} \oplus \Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\vartheta_{\mathbf{b}} \oplus \cdots \oplus \Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}/\boldsymbol{\lambda}-1}.$$

As in the proof of Proposition 4.22, we can identify $\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}$ with $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ using the isomorphism ρ_i, for $1 \le i \le p_{\boldsymbol{\lambda}}$. Then the highest weight vector $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ corresponds to the vector $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\Big(\prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}}\big(\vartheta_{\boldsymbol{\lambda}} - \mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{o_{\boldsymbol{\lambda}}t}\big)\Big)$. This implies that $\vartheta_{\boldsymbol{\lambda}} = \vartheta_{\mathbf{b}}^{p_{\mathbf{b}}/\boldsymbol{\lambda}}$ acts as the scalar $\mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{io_{\boldsymbol{\lambda}}}$ on the highest weight vector of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda}) \hookrightarrow \Delta^{\boldsymbol{\lambda}}_{i,p}$. All of the claims in the Corollary now follow. □

Corollary 4.24. *Suppose that $\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$.*

a) *If $1 \le i \le p_{\boldsymbol{\lambda}}$ then $L^{\boldsymbol{\lambda}}_{i,p}$ is the simple head of $\Delta^{\boldsymbol{\lambda}}_{i,p}$.*

b) *If $1 \le i \le p_{\boldsymbol{\lambda}}$ and $1 \le j \le p_{\boldsymbol{\mu}}$ then*

$$[\Delta^{\boldsymbol{\lambda}}_{i,p} : L^{\boldsymbol{\mu}}_{j,p}] = \begin{cases} \delta_{ij}, & \text{if } \boldsymbol{\lambda} = \boldsymbol{\mu},\\ 0, & \text{if } \boldsymbol{\lambda} \not\trianglerighteq \boldsymbol{\mu}.\end{cases}$$

Proof. By (4.14) $L_{\mathbf{b}}(\boldsymbol{\lambda})$ is the simple head of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ and

$$[\Delta_{\mathbf{b}}(\boldsymbol{\lambda}) : L_{\mathbf{b}}(\boldsymbol{\mu})] = \begin{cases} 1, & \text{if } \boldsymbol{\mu} = \boldsymbol{\lambda},\\ 0, & \text{if } \boldsymbol{\lambda} \not\trianglerighteq \boldsymbol{\mu}.\end{cases}$$

Hence, the result follows from Proposition 4.22 and Frobenius reciprocity. □

Recall that $\sim_{\mathbf{b}}$ is the equivalence relation on $\mathscr{P}_{d,\mathbf{b}}$ such that $\boldsymbol{\lambda} \sim_{\mathbf{b}} \boldsymbol{\mu}$ if $\boldsymbol{\mu} = \boldsymbol{\lambda}\langle ko_{\mathbf{b}}\rangle$ for some $k \in \mathbb{Z}$.

Corollary 4.25. *The algebra $\mathscr{S}_{r,p,n}(\mathbf{b})$ is split over K and*

$$\{\, L^{\boldsymbol{\lambda}}_{i,p} \mid \boldsymbol{\lambda} \in \mathscr{P}^{\mathbf{b}}_{d,\mathbf{b}} \text{ and } 1 \le i \le p_{\boldsymbol{\lambda}} \,\}$$

is a complete set of pairwise non-isomorphic absolutely irreducible $\mathscr{S}_{r,p,n}(\mathbf{b})$-modules.

Proof. Just as in the proof of Theorem 3.50, this follows from Corollary 4.24, Frobenius reciprocity and some general arguments in Clifford theory. □

Recall from (4.17) that the Schur functor $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}} : \text{Mod-}\mathscr{S}_{d,\mathbf{b}} \longrightarrow \text{Mod-}\mathscr{H}_{d,\mathbf{b}}$ is given by $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}(M) = M\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$, where $\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$ is the identity map on $\mathscr{H}_{d,\mathbf{b}}$. Using the embedding $\mathscr{S}_{d,\mathbf{b}} \hookrightarrow \mathscr{S}_{r,p,n}(\mathbf{b})$, and the fact that $v_{\mathbf{b}} = v_{\mathbf{b}}^{+}u_{\boldsymbol{\omega}_{\mathbf{b}}}^{+}$, it is easy to check that $\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$ corresponds to the natural projection from $\bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}} M^{\boldsymbol{\lambda}}_{\mathbf{b}}$ onto $V_{\mathbf{b}} = M^{\boldsymbol{\omega}_{\mathbf{b}}}_{\mathbf{b}}$. In particular,

$$\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}\mathscr{S}_{r,p,n}(\mathbf{b})\varphi_{\boldsymbol{\omega}_{\mathbf{b}}} = \mathscr{E}_{d,\mathbf{b}} \quad\text{and}\quad \varphi_{\boldsymbol{\omega}_{\mathbf{b}}}\mathscr{S}_{d,\mathbf{b}}\varphi_{\boldsymbol{\omega}_{\mathbf{b}}} = \mathscr{H}_{d,\mathbf{b}}.$$

Hence, we have a second Schur functor $\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}\colon \text{Mod-}\mathscr{S}_{r,p,n}(\mathbf{b})\longrightarrow\text{Mod-}\mathscr{E}_{d,\mathbf{b}}$ which is given by $\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}(M)=M\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$ and if $\varphi\in\operatorname{Hom}_{\mathscr{S}_{r,p,n}(\mathbf{b})}(M,N)$ then $\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}(\varphi)(x\varphi_{\boldsymbol{\omega}_{\mathbf{b}}})=\varphi(x)$, for all $x\in M$. It is straightforward to check that we have the following commutative diagram of functors:

$$\begin{tikzcd}
\text{Mod-}\mathscr{S}_{r,p,n}(\mathbf{b}) \arrow[r, "?\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}"] \arrow[d, "\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}"'] & \text{Mod-}\mathscr{S}_{d,\mathbf{b}} \arrow[d, "\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}"] \\
\text{Mod-}\mathscr{E}_{d,\mathbf{b}} \arrow[r, "?\downarrow^{\mathscr{E}_{d,\mathbf{b}}}_{\mathscr{H}_{d,\mathbf{b}}}"'] & \text{Mod-}\mathscr{H}_{d,\mathbf{b}}
\end{tikzcd}\tag{4.26}$$

Lemma 4.27. *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ and $1\le i\le p_{\boldsymbol{\lambda}}$. Then*

$$\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}(\Delta^{\boldsymbol{\lambda}}_{i,p})\cong S^{\boldsymbol{\lambda}}_{i,p}\quad\textit{and}\quad \mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}(L^{\boldsymbol{\lambda}}_{i,p})\cong\begin{cases}D^{\boldsymbol{\lambda}}_{i,p}, & \textit{if } \boldsymbol{\lambda}\in\mathscr{K}_{d,\mathbf{b}},\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. This follow directly from (4.26) and Lemma 4.7. □

Corollary 4.28. *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$, $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, $\boldsymbol{\mu}\in\mathscr{K}_{d,\mathbf{b}}$, $1\le i\le p_{\boldsymbol{\lambda}}$ and that $1\le j\le p_{\boldsymbol{\mu}}$. Then $[\Delta^{\boldsymbol{\lambda}}_{i,p}:L^{\boldsymbol{\mu}}_{j,p}]=[S^{\boldsymbol{\lambda}}_{i,p}:D^{\boldsymbol{\mu}}_{j,p}]=[S^{\boldsymbol{\lambda}}_{i}:D^{\boldsymbol{\mu}}_{j}]$.*

Proof. This follows directly from Lemma 4.27 and Lemma 4.7 together with the easily checked fact that the functors $\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}$ and $\mathtt{F}_{\mathscr{E}}$ are exact. □

Therefore, in order to compute the decomposition number $[S^{\boldsymbol{\lambda}}_{i}:D^{\boldsymbol{\mu}}_{j}]$ it is enough to compute the decomposition number $[\Delta^{\boldsymbol{\lambda}}_{i,p}:L^{\boldsymbol{\mu}}_{j,p}]$ for $\mathscr{S}_{r,p,n}$.

4.4. **Splittable decomposition numbers.** In this section we derive explicit formulae for the l-splittable decomposition numbers of the algebras $\mathscr{S}_{r,p,n}(\mathbf{b})$, and hence of $\mathscr{H}_{r,p,n}$ by Corollary 4.28, in characteristic zero. These decomposition numbers depend explicitly on the decomposition numbers of certain Ariki–Koike algebras and on the scalars $\mathfrak{g}_{\boldsymbol{\lambda}}$ introduced in Lemma 3.45. By Theorem 2.4 this will determine all of the decomposition numbers of $\mathscr{H}_{r,p,n}$.

Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions in $\mathscr{P}_{d,\mathbf{b}}$. We want to compute the decomposition numbers $[\Delta^{\boldsymbol{\lambda}}_{i,p}:L^{\boldsymbol{\mu}}_{j,p}]$ for $1\le i\le p_{\boldsymbol{\lambda}}$ and $1\le j\le p_{\boldsymbol{\mu}}$. By Corollary 4.23 and the exactness of $\vartheta_{\mathbf{b}}$, if $p_{\boldsymbol{\lambda}}=p_{\boldsymbol{\mu}}$ then

$$[\Delta^{\boldsymbol{\lambda}}_{i,p}:L^{\boldsymbol{\mu}}_{j,p}]=[\Delta^{\boldsymbol{\lambda}}_{i+1,p}:L^{\boldsymbol{\mu}}_{j+1,p}],\tag{4.29}$$

where we read $i+1$ and $j+1$ modulo $p_{\boldsymbol{\lambda}}$. Therefore, these decomposition numbers are determined by the decomposition numbers

$$d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}=[\Delta^{\boldsymbol{\lambda}}_{0,p}:L^{\boldsymbol{\mu}}_{j,p}],$$

for $1\le j\le p_{\boldsymbol{\mu}}$. In fact, as noted above, it is enough to compute the splittable decomposition numbers. That is, the $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ such that $p_{\boldsymbol{\lambda}}=p_{\boldsymbol{\mu}}$, for $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$.

Before we start to compute the decomposition numbers $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ we introduce some new notation. If A is any finite dimensional algebra let $\mathcal{R}(A)$ be the Grothendieck group of finitely generated A-modules. If M is an A-module let $[M]$ be the image of M in $\mathcal{R}(A)$. In particular, note that the Grothendieck group of $\mathcal{R}(\mathscr{S}_{r,n})$ is equipped with two distinguished bases:

$$\{\,[\Delta(\boldsymbol{\lambda})]\mid\boldsymbol{\lambda}\in\mathscr{P}_{r,n}\,\}\quad\text{and}\quad\{\,[L(\boldsymbol{\lambda})\mid\boldsymbol{\lambda}\in\mathscr{P}_{r,n}\,\}.$$

Similar remarks apply to the Grothendieck groups of the cyclotomic Schur algebras $\mathscr{S}_{d,\mathbf{b}}$ and $\mathscr{S}_{r,p,n}(\mathbf{b})$, for $\mathbf{b}\in\mathscr{C}_{p,n}$.

Fix integers l and m such that $p=lm$ and suppose that $\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$, for some $\mathbf{b}\in\mathscr{C}_{p,n}$. Then a multipartition $\boldsymbol{\mu}$ is **l-symmetric** if

$$\boldsymbol{\mu}=\boldsymbol{\nu}^{\,l}:=(\underbrace{\boldsymbol{\nu},\dots,\boldsymbol{\nu}}_{l\text{ times}}),$$

for some multipartition $\boldsymbol{\nu}\in\mathscr{P}_{r/l,n/l}$. Note that if $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is an l-splittable decomposition number then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are both l-symmetric multipartitions.

Let $\mathscr{P}^{\,l}_{d,\mathbf{b}}$ be the set of l-symmetric multipartitions in $\mathscr{P}_{d,\mathbf{b}}$. It is easy to see that

$$\mathscr{P}^{\,l}_{d,\mathbf{b}}=\{\,\boldsymbol{\mu}\mid\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}\text{ and }\mathsf{o}_{\boldsymbol{\mu}}|m\,\}.$$

If $\mathscr{P}^{\,l}_{d,\mathbf{b}}$ is non-empty then $\mathsf{o}_{\mathbf{b}}|m$ and we define $\mathbf{b}_m=(b_1,\dots,b_m)$. If $\boldsymbol{\mu}\in\mathscr{P}^{\,l}_{d,\mathbf{b}}$ define $\boldsymbol{\mu}_m=(\boldsymbol{\mu}^{[1]},\dots,\boldsymbol{\mu}^{[m]})$. Then $\boldsymbol{\mu}_m\in\mathscr{P}_{r/l,\mathbf{b}_m}\subseteq\mathscr{P}_{r/l,n/l}$. It is easy to check that the map $\boldsymbol{\nu}\mapsto\boldsymbol{\nu}^{\,l}$ defines a bijection from $\mathscr{P}_{r/l,\mathbf{b}_m}$ to $\mathscr{P}^{\,l}_{d,\mathbf{b}}$, with the inverse map being given by $\boldsymbol{\mu}\mapsto\boldsymbol{\mu}_m$.

We now return to our main task of computing splittable decomposition numbers. We will do this by deriving a system of equations which uniquely determine the decomposition numbers $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$, for $1\le j\le l=p_{\boldsymbol{\lambda}}$.

For the rest of this subsection fix $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ and set $m=\mathsf{o}_{\boldsymbol{\lambda}}$ and $l=p_{\boldsymbol{\lambda}}$. Then $\mathbf{b}_m=(b_1,\dots,b_m)\in\mathscr{C}_{r/l,n/l}$ and $\boldsymbol{\lambda}_m\in\mathscr{P}_{r/l,\mathbf{b}_m}$. By (4.9) the cyclotomic Schur algebras $\mathscr{S}_{r/l,\mathbf{b}_m}$ and $\mathscr{S}_{d,\mathbf{b}}$ are related by

$$\mathscr{S}_{r/l,\mathbf{b}_m}\cong\mathscr{S}_{d,b_1}\otimes\dots\otimes\mathscr{S}_{d,b_m}\quad\text{and}\quad\mathscr{S}_{d,\mathbf{b}}\cong\big(\mathscr{S}_{r/l,\mathbf{b}_m}\big)^{\otimes l}.$$

For $\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$ let $d_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m}=[\Delta_{\mathbf{b}_m}(\boldsymbol{\lambda}_m):L_{\mathbf{b}_m}(\boldsymbol{\mu}_m)]$ be the corresponding decomposition number for the cyclotomic Schur algebra $\mathscr{S}_{r/l,\mathbf{b}_m}$. Since

$$\Delta_{\mathbf{b}_m}(\boldsymbol{\lambda}_m)\cong\Delta(\boldsymbol{\lambda}^{[1]})\otimes\dots\otimes\Delta(\boldsymbol{\lambda}^{[m]})\quad\text{and}\quad L_{\mathbf{b}_m}(\boldsymbol{\mu}_m)\cong L(\boldsymbol{\mu}^{[1]})\otimes\dots\otimes L(\boldsymbol{\mu}^{[m]})$$

we have that

$$d_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m}=\prod_{i=1}^m[\Delta(\boldsymbol{\lambda}^{[i]}):L(\boldsymbol{\mu}^{[i]})]=d_{\boldsymbol{\lambda}^{[1]}\boldsymbol{\mu}^{[1]}}\dots d_{\boldsymbol{\lambda}^{[m]}\boldsymbol{\mu}^{[m]}},\tag{4.30}$$

where $d_{\boldsymbol{\lambda}^{[i]}\boldsymbol{\mu}^{[i]}}=[\Delta(\boldsymbol{\lambda}^{[i]}):L(\boldsymbol{\mu}^{[i]})]$, for $1\le i\le m=\mathsf{o}_{\boldsymbol{\lambda}}$.

Recall that if $\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$ then $p_{\mathbf{b}/\boldsymbol{\mu}}=p_{\mathbf{b}}/p_{\boldsymbol{\mu}}=\mathsf{o}_{\boldsymbol{\mu}}/\mathsf{o}_{\mathbf{b}}$. If $\boldsymbol{\mu}\in\mathscr{P}^{\,l}_{d,\mathbf{b}}$ is l-symmetric then $\mathsf{o}_{\boldsymbol{\mu}}$ divides m, so we define $p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}=p_{\boldsymbol{\mu}}/p_{\boldsymbol{\lambda}}$. Then $p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}\in\mathbb{N}$ and $p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}=\mathsf{o}_{\boldsymbol{\lambda}}/\mathsf{o}_{\boldsymbol{\mu}}=p_{\mathbf{b}/\boldsymbol{\lambda}}/p_{\mathbf{b}/\boldsymbol{\mu}}$.

Lemma 4.31. *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, $l=p_{\boldsymbol{\lambda}}$ and $m=\mathsf{o}_{\boldsymbol{\lambda}}$. Then:*

a) $[\Delta_{\mathbf{b}_m}(\boldsymbol{\lambda}_m)]=\displaystyle\sum_{\boldsymbol{\nu}\in\mathscr{P}^{\,l}_{d,\mathbf{b}}}d_{\boldsymbol{\lambda}_m\boldsymbol{\nu}_m}[L_{\mathbf{b}_m}(\boldsymbol{\nu}_m)]$.

b) $[\Delta^{\boldsymbol{\lambda}}_{0,p}]=\displaystyle\sum_{\boldsymbol{\nu}\in\mathscr{P}_{d,\mathbf{b}}}\sum_{1\le j\le p_{\boldsymbol{\nu}}}d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\nu}}[L^{\boldsymbol{\nu}}_{j,p}]$.

c) *If $\boldsymbol{\mu}\in\mathscr{P}^{\,l}_{d,\mathbf{b}}$ then $d^{(1)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}+d^{(2)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}+\dots+d^{(l)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}=p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}d^{\,l}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m}$.*

Proof. Part (a) is just a rephrasing of the definition of decomposition numbers combined with the bijection $\mathscr{P}^{\,l}_{d,\mathbf{b}}\xrightarrow{\simeq}\mathscr{P}_{r/l,\mathbf{b}_m};\boldsymbol{\mu}\mapsto\boldsymbol{\mu}_m$. Part (b) follows similarly.

Suppose that $\boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$. We prove (c) by computing the decomposition multiplicity of $L_{\mathbf{b}}(\boldsymbol{\mu})$ on both sides of part (b) upon restriction to $\mathscr{S}_{d,\mathbf{b}}$. By Corollary 4.23,

$$\Delta^{\boldsymbol{\lambda}}_{0,p}\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} \cong \Delta_{\mathbf{b}}(\boldsymbol{\lambda}) \oplus \Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{-1}} \oplus \cdots \oplus \Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{1-p_{\mathbf{b}/\boldsymbol{\lambda}}}}.$$

Now, every composition factor of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ is isomorphic to $L_{\mathbf{b}}(\boldsymbol{\nu})$, for some $\boldsymbol{\nu} \in \mathscr{P}_{d,\mathbf{b}}$, and $L_{\mathbf{b}}(\boldsymbol{\nu})^{\vartheta_{\mathbf{b}}} \cong L_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}(\boldsymbol{\nu})$ by Lemma 4.19. Therefore, the decomposition multiplicity of $L_{\mathbf{b}}(\boldsymbol{\mu})$ in $\Delta^{\boldsymbol{\lambda}}_{0,p}\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}$ is

$$\frac{p_{\mathbf{b}/\boldsymbol{\lambda}}}{p_{\mathbf{b}/\boldsymbol{\mu}}}[\Delta_{\mathbf{b}}(\boldsymbol{\lambda}) : L_{\mathbf{b}}(\boldsymbol{\mu})] = p_{\boldsymbol{\mu}/\boldsymbol{\lambda}} d^{l}_{\boldsymbol{\lambda}_m, \boldsymbol{\mu}_m},$$

where the second equality follows from (4.30).

Now consider the multiplicity of $L_{\mathbf{b}}(\boldsymbol{\mu})$ on the right hand side of (b). If $\boldsymbol{\nu} \in \mathscr{P}_{d,\mathbf{b}}$ and $1 \le j \le p_{\boldsymbol{\nu}}$ then, using Corollary 4.23 again,

$$L^{\boldsymbol{\lambda}}_{j,p}\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} \cong L_{\mathbf{b}}(\boldsymbol{\lambda}) \oplus L_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{-1}} \oplus \cdots \oplus L_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{1-p_{\mathbf{b}/\boldsymbol{\nu}}}}.$$

Therefore, $[L^{\boldsymbol{\lambda}}_{j,p}\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} : L_{\mathbf{b}}(\boldsymbol{\mu})] = 1$ by Lemma 4.19. Equating the multiplicity of $L_{\mathbf{b}}(\boldsymbol{\mu})$ on both sides of (b) now gives (c). □

Lemma 4.31 gives our first relation satisfied by the decomposition numbers $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$. We now use formal characters to find more relations. Let $K[\mathscr{P}_{r,n}]$ be the K-vector space with basis $\{\, e^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_{r,n} \,\}$. The ($K$-valued) **formal character** of the $\mathscr{S}_{d,\mathbf{b}}$-module M is

$$\operatorname{ch} M = \sum_{\boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}} (\dim M_{\boldsymbol{\mu}}) e^{\boldsymbol{\mu}},$$

an element of $K[\mathscr{P}_{r,n}]$. The coefficients appearing in the formal characters are the traces of the identity maps on the weight spaces. We need a more general version of the formal character which records the traces of powers of $\vartheta^t_{\boldsymbol{\lambda}}$ on certain weight spaces, for $1 \le t < l = p_{\boldsymbol{\lambda}}$.

Fix an integer t with $1 \le t < p_{\boldsymbol{\lambda}}$. Let $l_t = \gcd(t, l)$ be the greatest common divisor of t and l and set $\ell_t = l/l_t$. By convention, we set $l_0 = l$. Then $r/\ell_t = dml_t$ so that $K[\mathscr{P}_{dml_t, n/\ell_t}] = K[\mathscr{P}_{r/\ell_t, n/\ell_t}]$.

Now suppose that M is an $\mathscr{S}_{r,p,n}(\mathbf{b})$-module and that $\boldsymbol{\gamma} = \gamma^{\ell_t}$ is an ℓ_t-symmetric multipartition. We will show in Lemma 4.32 below that $\vartheta^t_{\boldsymbol{\lambda}}$ stabilizes each ℓ_t-symmetric weight space $M_{\gamma^{\ell_t}}$. With this in mind, we define the **twining character** of M to be

$$\operatorname{ch}^{\mathrm{l}}_{\mathrm{t}} M = \sum_{\gamma \in \mathscr{P}_{r/\ell_t, n/\ell_t}} \operatorname{Tr}\left(\vartheta^t_{\boldsymbol{\lambda}}, M_{\gamma^{\ell_t}}\right) e^{\gamma} \in K[\mathscr{P}_{r/\ell_t, n/\ell_t}].$$

It is easy to see that, just like the usual character, the twining character lifts to a well-defined map $\operatorname{ch}^{\mathrm{l}}_{\mathrm{t}} : \mathcal{R}(\mathscr{S}_{r,p,n}(\mathbf{b})) \longrightarrow K[\mathscr{P}_{r/\ell_t, n/\ell_t}]$ on the Grothendieck group of $\mathscr{S}_{r,p,n}(\mathbf{b})$.

The following Lemma will allow us to compute the twining character $\operatorname{ch}^{\mathrm{l}}_{\mathrm{t}}$ on both sides of Lemma 4.31(b).

Lemma 4.32. *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ and $1 \le t < l = p_{\boldsymbol{\lambda}}$. Then*

$$ch^{\mathrm{l}}_{\mathrm{t}} \Delta^{\boldsymbol{\lambda}}_{i,p} = \varepsilon^{itm} p_{\mathbf{b}/\boldsymbol{\lambda}} \mathfrak{g}^t_{\boldsymbol{\lambda}} \operatorname{ch} \Delta_{\mathbf{b}_{l_t m}}(\boldsymbol{\lambda}_{l_t m}),$$

for $1\le i\le p_{\boldsymbol\lambda}$. *Moreover, if* $\boldsymbol\mu\in\mathscr{P}^{\,l}_{d,\mathbf{b}}$ *and* $1\le j\le p_{\boldsymbol\mu}$ *then*

$$ch^{\mathrm{l}}_{\mathrm{t}} L^{\boldsymbol\mu}_{j,p}=\varepsilon^{jtm}p_{\mathbf{b}/\boldsymbol\mu}\mathfrak{g}_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}}\operatorname{ch}L_{\mathbf{b}_{l_tm}}(\boldsymbol\mu_{l_tm}).$$

Proof. We only prove the formula for $\operatorname{ch}^{\mathrm{l}}_{\mathrm{t}} L^{\boldsymbol\mu}_{j,p}$ and leave the almost identical calculation of $\operatorname{ch}^{\mathrm{l}}_{\mathrm{t}}\Delta^{\boldsymbol\lambda}_{0,p}$ to the reader. To ease the notation let $m'=\mathsf{o}_{\boldsymbol\mu}$ so that $\mathbf{b}_{m'}=(\mathbf{b}^{[1]},\dots,\mathbf{b}^{[m']})$ and $\boldsymbol\mu_{m'}=(\boldsymbol\mu^{[1]},\dots,\boldsymbol\mu^{[m']})\in\mathscr{P}_{r/p_{\boldsymbol\mu},\mathbf{b}_{m'}}$.

To determine $\operatorname{ch}^{\mathrm{l}}_{\mathrm{t}} L^{\boldsymbol\mu}_{j,p}$ for each $\gamma\in\mathscr{P}_{r/\ell_t,n/\ell_t}$ we need to compute

$$\begin{aligned}\operatorname{Tr}\big(\vartheta^t_{\boldsymbol\lambda},(L^{\boldsymbol\mu}_{j,p})_{\gamma^{\ell_t}}\big)&=\operatorname{Tr}\big(\vartheta_{\mathbf{b}}^{tp_{\mathbf{b}/\boldsymbol\lambda}},(L^{\boldsymbol\mu}_{j,p})_{\gamma^{\ell_t}}\big)=\operatorname{Tr}\big((\vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol\mu}})^{tp_{\boldsymbol\mu/\boldsymbol\lambda}},(L^{\boldsymbol\mu}_{j,p})_{\gamma^{\ell_t}}\big)\\&=\operatorname{Tr}\big(\vartheta_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}},(L^{\boldsymbol\mu}_{j,p})_{\gamma^{\ell_t}}\big).\end{aligned}$$

By Corollary 4.23 we can identify $L^{\boldsymbol\mu}_{j,p}$ with the K-vector space

$$L_{\mathbf{b}}(\boldsymbol\mu)\oplus L_{\mathbf{b}}(\boldsymbol\mu)\vartheta_{\mathbf{b}}\oplus\dots\oplus L_{\mathbf{b}}(\boldsymbol\mu)\vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol\mu}-1},$$

where the action of $\mathscr{S}_{r,p,n}(\mathbf{b})$ on $L^{\boldsymbol\mu}_{j,p}$ is determined by

a) $L^{\boldsymbol\mu}_{j,p}\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}\cong L_{\mathbf{b}}(\boldsymbol\mu)\oplus L_{\mathbf{b}}(\boldsymbol\mu)^{\vartheta_{\mathbf{b}}^{-1}}\oplus\dots\oplus L_{\mathbf{b}}(\boldsymbol\mu)^{\vartheta_{\mathbf{b}}^{1-p_{\mathbf{b}/\boldsymbol\mu}}}$,
b) $(x\vartheta^a_{\mathbf{b}})\vartheta^c_{\mathbf{b}}=x\vartheta^{a+c}_{\mathbf{b}}$, for all $x\in L_{\mathbf{b}}(\boldsymbol\mu)$ and $a,c\in\mathbb{Z}$,
c) $\vartheta_{\boldsymbol\mu}$ acts as the scalar $\varepsilon^{j\mathsf{o}_{\boldsymbol\mu}}\mathfrak{g}_{\boldsymbol\mu}$ on the highest weight vector of $L_{\mathbf{b}}(\boldsymbol\mu)$.

Note that $p_{\boldsymbol\mu/\boldsymbol\lambda}=m/m'\in\mathbb{N}$, since $\boldsymbol\mu\in\mathscr{P}^{\,l}_{d,\mathbf{b}}$, and $\vartheta_{\boldsymbol\mu}=\vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol\mu}}=\vartheta_{\boldsymbol\lambda}^{p_{\boldsymbol\lambda/\boldsymbol\mu}}$. Therefore,

$$\operatorname{Tr}\big(\vartheta^t_{\boldsymbol\lambda},(L^{\boldsymbol\mu}_{j,p})_{\gamma^{\ell_t}}\big)=\operatorname{Tr}\big(\vartheta_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}},(L^{\boldsymbol\mu}_{j,p_{\boldsymbol\mu}})_{\gamma^{\ell_t}}\big)=p_{\mathbf{b}/\boldsymbol\mu}\operatorname{Tr}\big(\vartheta_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}},L_{\mathbf{b}}(\boldsymbol\mu)_{\gamma^{\ell_t}}\big).$$

To compute this trace first observe that if $\overline{\varphi}_{\mathfrak{t}^{\boldsymbol\mu}}$ is the highest weight vector of $L_{\mathbf{b}}(\boldsymbol\mu)$ then, by (c) above (which comes from Corollary 4.23),

$$\overline{\varphi}_{\mathfrak{t}^{\boldsymbol\mu}}\vartheta^t_{\boldsymbol\lambda}=\varepsilon^{jtm}\mathfrak{g}_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}}\overline{\varphi}_{\mathfrak{t}^{\boldsymbol\mu}}.\tag{4.33}$$

Now, $p=\ell_tl_tm=\ell_tl_tp_{\boldsymbol\mu/\boldsymbol\lambda}m'$ so we can identify the two modules $L_{\mathbf{b}}(\boldsymbol\mu)$ and $L_{\mathbf{b}_{l_tm}}(\boldsymbol\mu_{l_tm})^{\otimes\ell_t}$. Using Lemma 4.15, if $1\le j\le p/\ell_t$ then

$$\varphi^{(j)}_{\mathsf{ST}}\vartheta^t_{\boldsymbol\lambda}=\varepsilon^{-mtk}\vartheta^t_{\boldsymbol\lambda}\varphi^{(tm+j)}_{\mathsf{ST}}\tag{4.34}$$

for some $k\in\mathbb{Z}$, where we identity $\varphi^{(j)}_{\mathsf{ST}}$ and $\varphi^{(j')}_{\mathsf{ST}}$ if $j\equiv j'\pmod p$. Therefore, since $\overline{\varphi}_{\mathfrak{t}^{\boldsymbol\mu}}$ generates $L_{\mathbf{b}}(\boldsymbol\mu)$, it follows from (4.33) and (4.34) that each simple p-tensor

$$\boldsymbol\beta=(x^{(1)}_1\otimes\dots\otimes x^{(1)}_{l_tm})\otimes\dots\otimes(x^{(\ell_t)}_1\otimes\dots\otimes x^{(\ell_t)}_{l_tm})$$

in $L_{\mathbf{b}}(\boldsymbol\mu)_{\gamma^{\ell_t}}$ is mapped by $\vartheta^t_{\boldsymbol\lambda}=\vartheta_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}}$ to a scalar multiple of

$$(x^{(tm+1)}_1\otimes\dots\otimes x^{(tm+1)}_{l_tm})\otimes\dots\otimes(x^{(tm+\ell_t)}_1\otimes\dots\otimes x^{(tm+\ell_t)}_{l_tm}),$$

where we identity $x^{(j)}_i=x^{(j')}_i$ whenever $j\equiv j'\pmod{\ell_t}$ for $1\le i\le l_tm$. Thus, to calculate $\operatorname{Tr}\big(\vartheta^t_{\boldsymbol\lambda},L_{\mathbf{b}}(\boldsymbol\mu)\big)$ we only need to consider the case when $x^{(s)}_i=x^{(tm+s)}_i$, for all $1\le i\le l_tm$ and all $1\le s\le\ell_t$. By construction, $(tm)/(l_tm)\not\equiv0\pmod{\ell_t}$, so this can only happen if

$$x^{(s)}_i=x^{(s')}_i,\quad\text{whenever }1\le i\le l_tm\text{ and }1\le s,s'\le\ell_t.$$

Consequently, $\boldsymbol\beta$ contributes to the twining character only if $\boldsymbol\beta=\beta\otimes\dots\otimes\beta$ (ℓ_t times), for some $\beta\in L_{\mathbf{b}_{l_tm}}(\boldsymbol\mu_{l_tm})$. Notice that if $\beta\in L_{\mathbf{b}_{l_tm}}(\boldsymbol\mu_{l_tm})_\gamma$, for some $\gamma\in\mathscr{P}_{r/\ell_t,n/\ell_t}$ then $\boldsymbol\beta\in L_{\mathbf{b}}(\boldsymbol\mu)_{\gamma^{\ell_t}}$. In particular, this shows that $\vartheta^t_{\boldsymbol\lambda}$ stabilizes $L_{\mathbf{b}_{l_tm}}(\boldsymbol\mu_{l_tm})_\gamma$ as we claimed when introducing the twining character.

In (4.33) we have already shown that $\vartheta^t_{\boldsymbol\lambda}$ acts as multiplication by $\varepsilon^{jtm}\mathfrak{g}_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}}$ on the highest weight vector of $L_{\mathbf{b}_{l_tm}}(\boldsymbol\mu_{l_tm})^{\otimes\ell_t}$. On the other hand, by (4.34) and abusing the notation of Lemma 4.15 slightly, if $1\le j\le\ell_t$ then

$$\big(\varphi^{(j)}_{\mathsf{ST}}\big)^{\otimes\ell_t}\vartheta^t_{\boldsymbol\lambda}=\varepsilon^{-mt\ell_tk}\vartheta^t_{\boldsymbol\lambda}\big(\varphi^{(j)}_{\mathsf{ST}}\big)^{\otimes\ell_t}=\vartheta^t_{\boldsymbol\lambda}\big(\varphi^{(j)}_{\mathsf{ST}}\big)^{\otimes\ell_t},$$

where the last equality follows because $mt\ell_t=p(t/l_t)$ is divisible by p. Therefore, writing $\beta^{\otimes\ell_t}=\overline{\varphi}_{\mathfrak{t}\boldsymbol\mu}\varphi^{\otimes\ell_t}$, for some $\varphi\in\mathscr{S}_{l_tm,\mathbf{b}_{l_tm}}$, we have that

$$\beta^{\otimes\ell_t}\vartheta^t_{\boldsymbol\lambda}=\overline{\varphi}_{\mathfrak{t}\boldsymbol\mu}\varphi^{\otimes\ell_t}\vartheta^t_{\boldsymbol\lambda}=\overline{\varphi}_{\mathfrak{t}\boldsymbol\mu}\vartheta^t_{\boldsymbol\lambda}\varphi^{\otimes\ell_t}=\varepsilon^{jtm}\mathfrak{g}_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}}\overline{\varphi}_{\mathfrak{t}\boldsymbol\mu}\varphi^{\otimes\ell_t}=\varepsilon^{jtm}\mathfrak{g}_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}}\beta^{\otimes\ell_t},$$

where the third equality uses (4.33). Consequently,

$$\operatorname{Tr}\big(\vartheta^t_{\boldsymbol\lambda},\big(L^{\boldsymbol\mu}_{j,p}\big)_{\gamma^{\ell_t}}\big)=p_{\mathbf{b}/\boldsymbol\mu}\varepsilon^{jtm}\mathfrak{g}_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}}\dim L_{\mathbf{b}_{l_tm}}(\boldsymbol\mu_{l_tm})_\gamma.$$

Summing over $\mathscr{P}^l_{d,\mathbf{b}}$ gives the desired formula for $\operatorname{ch}^{\mathrm{l}}_{\mathrm{t}}\big(L^{\boldsymbol\mu}_{j,p}\big)$ and completes the proof. □

Corollary 4.35. *Suppose that $\boldsymbol\lambda,\boldsymbol\mu\in\mathscr{P}^l_{d,\mathbf{b}}$, and $0\le t<l=p_{\boldsymbol\lambda}$, $l'=p_{\boldsymbol\mu}$. Then in K*

$$p_{\boldsymbol\mu/\boldsymbol\lambda}\Big(\frac{\mathfrak{g}_{\boldsymbol\lambda}}{\mathfrak{g}_{\boldsymbol\mu}^{p_{\boldsymbol\mu/\boldsymbol\lambda}}}\Big)^t d^{l_t}_{\boldsymbol\lambda_m,\boldsymbol\mu_m}=\varepsilon^{tm}d^{(1)}_{\boldsymbol\lambda\boldsymbol\mu}+\varepsilon^{2tm}d^{(2)}_{\boldsymbol\lambda\boldsymbol\mu}+\cdots+\varepsilon^{l'tm}d^{(l')}_{\boldsymbol\lambda\boldsymbol\mu}.$$

Proof. If $t=0$ then the result is just Lemma 4.31(c). If $t\ne1$ then combining Lemma 4.32 and Lemma 4.31(b) shows that

$$\operatorname{ch}\Delta_{\mathbf{b}_m}(\boldsymbol\lambda_m)^{\otimes l_t}=\sum_{\boldsymbol\mu\in\mathscr{P}^l_{d,\mathbf{b}}}\sum_{1\le j\le p_{\boldsymbol\mu}}\varepsilon^{jmt}d^{(j)}_{\boldsymbol\lambda\boldsymbol\mu}\frac{p_{\mathbf{b}/\boldsymbol\mu}\mathfrak{g}_{\boldsymbol\mu}^{tp_{\boldsymbol\mu/\boldsymbol\lambda}}}{p_{\mathbf{b}/\boldsymbol\lambda}\mathfrak{g}^t_{\boldsymbol\lambda}}\operatorname{ch}L_{\mathbf{b}_m}(\boldsymbol\mu_m)^{\otimes l_t}.$$

On the other hand, by Lemma 4.31(a),

$$\operatorname{ch}\Delta_{\mathbf{b}_m}(\boldsymbol\lambda_m)^{\otimes l_t}=\sum_{\boldsymbol\mu\in\mathscr{P}^l_{d,\mathbf{b}}}d^{l_t}_{\boldsymbol\lambda_m\boldsymbol\mu_m}\operatorname{ch}L_{\mathbf{b}_m}(\boldsymbol\mu_m)^{\otimes l_t}.$$

As the characters $\{\operatorname{ch}L_{\mathbf{b}_m}(\boldsymbol\nu_m)\}$ are linearly independent, comparing the coefficient of $\operatorname{ch}L_{\mathbf{b}_m}(\boldsymbol\mu_m)$ on both sides gives the result. □

Corollary 4.36. *Suppose that l divides p, $\boldsymbol\lambda,\boldsymbol\mu\in\mathscr{P}^l_{d,\mathbf{b}}$ and that $p_{\boldsymbol\lambda}=l=p_{\boldsymbol\mu}$. If $0\le t<l$ then, in K,*

$$\Big(\frac{\mathfrak{g}_{\boldsymbol\lambda}}{\mathfrak{g}_{\boldsymbol\mu}}\Big)^t d^{l_t}_{\boldsymbol\lambda_m\boldsymbol\mu_m}=\varepsilon^{tm}d^{(1)}_{\boldsymbol\lambda\boldsymbol\mu}+\varepsilon^{2tm}d^{(2)}_{\boldsymbol\lambda\boldsymbol\mu}+\cdots+\varepsilon^{ltm}d^{(l)}_{\boldsymbol\lambda\boldsymbol\mu}.$$

We can now complete the proof of the main results of this paper. Recall from just before Theorem D in the introduction that we defined matrices $V(l)$ and $V_i(l)$, whenever l divides p and $1\le i\le l$.

Theorem 4.37. *Suppose that $\boldsymbol\lambda,\boldsymbol\mu\in\mathscr{P}_{d,\mathbf{b}}$ and $p_\lambda=l=p_{\boldsymbol\lambda}$, for some $\mathbf{b}\in\mathscr{C}_{p,n}$. Then, for $1\le j\le p_{\boldsymbol\lambda}$,*

$$[\Delta^{\boldsymbol\lambda}_{0,p}:L^{\boldsymbol\mu}_{j,p}]\equiv\frac{\det V_j(l)}{\det V(l)}\pmod{\operatorname{char}K}.$$

In particular, $[\Delta^{\boldsymbol\lambda}_{0,p}:L^{\boldsymbol\mu}_{j,p}]=\frac{\det V_j(l)}{\det V(l)}$ if K is a field of characteristic zero.

Proof. By Corollary 4.36 the decomposition numbers $d^{(1)}_{\boldsymbol\lambda\boldsymbol\mu},\dots,d^{(l)}_{\boldsymbol\lambda\boldsymbol\mu}$ satisfy the matrix equation

$$V(l)\begin{pmatrix} d^{(1)}_{\boldsymbol\lambda\boldsymbol\mu}\\ \vdots\\ d^{(l)}_{\boldsymbol\lambda_m\boldsymbol\mu_m}\end{pmatrix}=\begin{pmatrix}\left(\frac{\mathfrak{g}_{\boldsymbol\lambda}}{\mathfrak{g}_{\boldsymbol\mu}}\right)^0 d^{l_0}_{\boldsymbol\lambda_m\boldsymbol\mu_m}\\ \vdots\\ \left(\frac{\mathfrak{g}_{\boldsymbol\lambda}}{\mathfrak{g}_{\boldsymbol\mu}}\right)^{l-1} d^{l_{(l-1)}}_{\boldsymbol\lambda_m,\boldsymbol\mu_m}\end{pmatrix}$$

Hence, the theorem follows by Cramer's rule. □

Observe that the condition $p_{\boldsymbol\lambda}=l=p_{\boldsymbol\mu}$ says that the decomposition numbers $[\Delta^{\boldsymbol\lambda}_{i,p}:L^{\boldsymbol\mu}_{j,p}]$ are l-splittable, for $1\le i,j<l$. Moreover, $[\Delta^{\boldsymbol\lambda}_{i,p}:L^{\boldsymbol\mu}_{j,p}]=[\Delta^{\boldsymbol\lambda}_{0,p}:L^{\boldsymbol\mu}_{j-i,p}]$ by (4.29). Hence, by Corollary 4.28 and Theorem 4.37 we have computed all of the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}(\mathbf{b})$ and $\mathscr{H}_{r,p,n}$.

Corollary 4.38. *Suppose that $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf{b}}$, $\boldsymbol\mu\in\mathscr{K}_{d,\mathbf{b}}$, for some $\mathbf{b}\in\mathscr{C}_{p,n}$, and that $p_{\boldsymbol\lambda}=p_{\boldsymbol\mu}=l$. Then, for $1\le i,j\le p_{\boldsymbol\lambda}$,*

$$[S^{\boldsymbol\lambda}_i:D^{\boldsymbol\mu}_j]=[\Delta^{\boldsymbol\lambda}_{i,p}:L^{\boldsymbol\mu}_{j,p}]\equiv\frac{\det V_{j-i}(l)}{\det V(l)}\pmod{\operatorname{char}K}.$$

In particular, this establishes Theorem D from the introduction. Finally, we are able to prove Theorem A, our Main Theorem from the introduction.

of Theorem A. By Theorem 2.4 the decomposition numbers of $\mathscr{H}_{r,p,n}$ are completely determined by the l-splittable decomposition numbers of the Hecke algebras $\mathscr{H}_{s,l,m}$, where l divides p, $1\le s\le r$ and $1\le m\le n$. Hence, Theorem A follows from Corollary 4.38. □

We remind the reader that the polynomials $\dot{\mathfrak{g}}_{\boldsymbol\lambda}$, for $\boldsymbol\lambda\in\mathscr{P}_{,\mathbf{b}}$, are determined by Proposition 3.40 and Remark 3.41. Hence, this result explicitly determines the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}$ (and of $\mathscr{H}_{r,p,n}$).

When K is a field of positive characteristic the results above only determine the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}$ and $\mathscr{H}_{r,p,n}$ modulo the characteristic of K.

Appendix A. Technical calculations for $v_{\mathbf{b}}$

In Chapter 2 we omitted the proofs of Propositions 2.13 and 2.17 and Lemma 2.34 because their proofs are long and uninspiring calculations. This appendix proves these three results.

A1. **Proof of Proposition 2.13.** We start by proving Proposition 2.13, which gives several different expressions for the element $v_{\mathbf{b}}$ from Definition 2.12.

We need the following fact which is generalisation of a fundamental result of Dipper and James [10, Lemma 3.10].

Lemma A1 ([12, Proposition 3.4])). *Suppose that a, b, s and t are positive integers with $1\le a+b<n$ and $1\le s\le t\le p$. Let $v^{(s,t)}_{a,b}=\mathcal{L}^{(s,t)}_{1,a}T_{a,b}\mathcal{L}^{(t+1,s-1)}_{1,b}$. Then*

$$T_i v^{(s,t)}_{a,b}=v^{(s,t)}_{a,b}T_{(i)w_{a,b}}\qquad\text{and}\qquad L_j v^{(s,t)}_{a,b}=v^{(s,t)}_{a,b}L_{(j)w_{a,b}},$$

for all i,j such that $1\le i,j\le a+b$ and $i\ne a,a+b$.

Recall from Section 2.1 that $\mathbf{b}\langle k\rangle=(b_{k+1},b_{k+2},\dots,b_{k+p})$ if $\mathbf{b}\in\mathscr{C}_{p,n}$ and $k\in\mathbb{Z}$, where we set $b_{i+p}=b_i$ for $1\le i\le p$.

Lemma A2. *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$ *and that* $1 \le j \le s \le p$. *Then*

$$\prod_{j\le k<s} \mathcal{L}^{(j,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_j} \cdot \prod_{j<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}$$
$$= \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \mathcal{L}^{(j)}_{1,\mathbf{b}^s_{j+1}} T_{\mathbf{b}^s_{j+1},b_j} \prod_{1\le i<j} \mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}} \cdot \prod_{j<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j},$$

where all products are read from left to right with <u>*decreasing*</u> *values of* i *and* k.

Proof. Let $L(s)$ and $R(s)$, respectively, be the left and right hand side of the formula in the statement of the Lemma. We show that $L(s) = R(s)$ by induction on s. To start the induction observe that, by our conventions,

$$L(j) = \prod_{j<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}} = R(j).$$

Hence, the Lemma is true when $s = j$. If $j \le s < p$ then, by induction,

$$\begin{aligned}
L(s+1) &= \mathcal{L}^{(j,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}^s_j} L(s) = \mathcal{L}^{(j,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}^s_j} R(s)\\
&= \mathcal{L}^{(j,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}^s_j} \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \mathcal{L}^{(j)}_{1,\mathbf{b}^s_{j+1}} T_{\mathbf{b}^s_{j+1},b_j} \prod_{1\le i<j} \mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}} \cdot \prod_{j<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j}\\
&= \mathcal{L}^{(j,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}^s_j} \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times v^{(1,j)}_{\mathbf{b}^s_{j+1},b_j} \prod_{j+1<k\le p} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j},
\end{aligned}$$

since $T_{a,b}$ commutes with $\mathcal{L}^{(i)}_{1,k}$ by Lemma 2.8 whenever $a+b \le k$ and $1 \le i \le p$ (we use this fact several times below). Therefore, using Lemma 2.11 and Lemma A1,

$$\begin{aligned}
L(s+1) &= \mathcal{L}^{(j,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}^s_{j+1}} T^{\langle \mathbf{b}^s_{j+1}\rangle}_{b_{s+1},b_j} \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}^s_{j+1}} v^{(1,j)}_{\mathbf{b}^s_{j+1},b_j} \prod_{j+1<k\le s} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \cdot \prod_{s+1<k\le p} \mathcal{L}^{(k)}_{\mathbf{b}^s_j+1,\mathbf{b}^{k-1}_j}\\
&= \mathcal{L}^{(j)}_{1,b_{s+1}} v^{(j+1,s)}_{b_{s+1},\mathbf{b}^s_{j+1}} \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times T^{\langle \mathbf{b}^s_{j+1}\rangle}_{b_{s+1},b_j} T_{\mathbf{b}^s_{j+1},b_j} \mathcal{L}^{(j+1,p)}_{1,b_j} \prod_{j+1<k\le s} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \cdot \prod_{s+1<k\le p} \mathcal{L}^{(k)}_{\mathbf{b}^s_j+1,\mathbf{b}^{k-1}_j}\\
&= v^{(j+1,s)}_{b_{s+1},\mathbf{b}^s_{j+1}} \mathcal{L}^{(j)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^{s+1}_{j+1}} \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times T_{\mathbf{b}^{s+1}_{j+1},b_j} \mathcal{L}^{(j+1,p)}_{1,b_j} \prod_{j+1<k\le s} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \cdot \prod_{s+1<k\le p} \mathcal{L}^{(k)}_{\mathbf{b}^s_j+1,\mathbf{b}^{k-1}_j}\\
&= \prod_{j+1\le k<s+1} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^{s+1}_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}^s_{j+1}} v^{(1,j)}_{\mathbf{b}^{s+1}_{j+1},b_j} \prod_{j+1<k\le s} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \cdot \prod_{s+1<k\le p} \mathcal{L}^{(k)}_{\mathbf{b}^s_j+1,\mathbf{b}^{k-1}_j}\\
&= R(s+1),
\end{aligned}$$

where the two lines we have, in essence, reversed some of the previous steps. This completes the proof. □

We are now ready to prove Proposition 2.13. This result includes the definition of $v_{\mathbf{b}}$ as the special case $j=1$. For the reader's convenience we restate the result.

Proposition A3. *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$ and $1\le j\le p$. Then*

$$v_{\mathbf{b}}=\prod_{j\le k<p}\mathcal{L}^{(j,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_j}\cdot\prod_{1\le i<j}\mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}\cdot\prod_{j<k\le p}\mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j}\cdot\prod_{1<i\le j}T_{\mathbf{b}^p_i,b_{i-1}}\mathcal{L}^{(i,p)}_{1,b_{i-1}},$$

where all products are read from left to right with <u>*decreasing*</u> *values of i and k.*

Proof. We argue by induction on j. When $j=1$ the Lemma is a restatement of Definition 2.12, so there is nothing to prove. Suppose now that $1\le j<p$ and that the formula in the Proposition holds. Then by induction and Lemma A2 (with $s=p$), we see that

$$\begin{aligned}
v_{\mathbf{b}}&=\prod_{j+1\le k<p}\mathcal{L}^{(j,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_j}\cdot\prod_{1\le i<j}\mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}\cdot\prod_{j<k\le p}\mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j}\cdot\prod_{1<i\le j}T_{\mathbf{b}^p_i,b_{i-1}}\mathcal{L}^{(i,p)}_{1,b_{i-1}}\\
&=\prod_{j+1\le k<s}\mathcal{L}^{(j+1,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_{j+1}}\cdot\mathcal{L}^{(j)}_{1,\mathbf{b}^p_{j+1}}T_{b^p_{j+1},b_j}\prod_{1\le i<j}\mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}\cdot\prod_{j<k\le p}\mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j}\\
&\qquad\times\prod_{1<i\le j}T_{\mathbf{b}^p_i,b_{i-1}}\mathcal{L}^{(i,p)}_{1,b_{i-1}}\\
&=\prod_{j+1\le k<p}\mathcal{L}^{(j+1,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_{j+1}}\cdot\prod_{1\le i<j}\mathcal{L}^{(i)}_{\mathbf{b}^p_{j+1}+1,\mathbf{b}^p_{i+1}}\cdot v^{(1,j)}_{\mathbf{b}^p_{j+1},b_j}\cdot\prod_{j+1<k\le p}\mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j}\\
&\qquad\times\prod_{1<i\le j}T_{\mathbf{b}^p_i,b_{i-1}}\mathcal{L}^{(i,p)}_{1,b_{i-1}}\\
&=\prod_{j+1\le k<p}\mathcal{L}^{(j+1,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_{j+1}}\cdot\prod_{1\le i<j}\mathcal{L}^{(i)}_{\mathbf{b}^p_{j+1}+1,\mathbf{b}^p_{i+1}}\cdot\prod_{j+1<k\le p}\mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_{j+1}}\\
&\qquad\times v^{(1,j)}_{\mathbf{b}^p_{j+1},b_j}\prod_{1<i\le j}T_{\mathbf{b}^p_i,b_{i-1}}\mathcal{L}^{(i,p)}_{1,b_{i-1}}\\
&=\prod_{j+1\le k<p}\mathcal{L}^{(j+1,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_{j+1}}\cdot\prod_{1\le i<j+1}\mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}\cdot\prod_{j+1<k\le p}\mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_{j+1}}\cdot\prod_{1<i\le j+1}T_{\mathbf{b}^p_i,b_{i-1}}\mathcal{L}^{(i,p)}_{1,b_{i-1}},
\end{aligned}$$

which is precisely the statement of the Proposition for $j+1$. □

A2. **Proof of Proposition 2.17.** Proposition 2.17 is quite an important result because it implies the existence of the central element $z_{\mathbf{b}}\in\mathscr{H}[d,\mathbf{b}]$. See the proof of Lemma 2.25.

Proposition A4. *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$. Then $Y_pY_{p-1}\dots Y_2Y_1=v_{\mathbf{b}}T_{\mathbf{b}}$.*

Proof. To prove the Lemma it is enough to show by induction on t that

$$Y_t\dots Y_1=\mathcal{L}^{(1,t-1)}_{1,b_t}T_{b_t,\mathbf{b}^{t-1}_1}\dots\mathcal{L}^{(1,1)}_{1,\mathbf{b}_2}T_{b_2,\mathbf{b}^1_1}\mathcal{L}^{(2)}_{1,b_1}\dots\mathcal{L}^{(t)}_{1,\mathbf{b}^{t-1}_1}\prod_{t<s\le p}\mathcal{L}^{(s)}_{1,\mathbf{b}^t_1}\cdot T^{\langle\mathbf{b}^{t-1}_1\rangle}_{b_t,\mathbf{b}^p_{t+1}}\dots T_{b_1,\mathbf{b}^p_2}.$$

When $t = 1$ the right hand side of this equation is just Y_1 so there is nothing to prove. Now suppose that $1 < t < p-1$. Then, by induction and Lemma 2.9,

$$\begin{aligned}
Y_{t+1}\dots Y_1 &= \mathcal{L}^{(t+2,t+p)}_{1,b_{t+1}} T_{b_{t+1},n-b_{t+1}} \cdot \mathcal{L}^{(1,t-1)}_{1,b_t} T_{b_t,\mathbf{b}_1^{t-1}} \dots \mathcal{L}^{(1,1)}_{1,\mathbf{b}_2} T_{b_2,\mathbf{b}_1^1} \\
&\qquad \times \mathcal{L}^{(2)}_{1,b_1} \dots \mathcal{L}^{(t)}_{1,\mathbf{b}_1^{t-1}} \prod_{t<s\le p} \mathcal{L}^{(s)}_{1,\mathbf{b}_1^t} \cdot T^{\langle \mathbf{b}_1^{t-1}\rangle}_{b_t,\mathbf{b}^p_{t+1}} \dots T^{\langle \mathbf{b}_1^1\rangle}_{b_2,\mathbf{b}_3^p} T_{b_1,\mathbf{b}_2^p} \\
&= \mathcal{L}^{(t+2,t+p)}_{1,b_{t+1}} \cdot T_{b_{t+1},\mathbf{b}_1^t} T^{\langle \mathbf{b}_1^t\rangle}_{b_{t+1},\mathbf{b}^p_{t+2}} \cdot \mathcal{L}^{(1,t-1)}_{1,b_t} T_{b_t,\mathbf{b}_1^{t-1}} \dots \mathcal{L}^{(1,1)}_{1,\mathbf{b}_2} T_{b_2,\mathbf{b}_1^1} \\
&\qquad \times \mathcal{L}^{(2)}_{1,b_1} \dots \mathcal{L}^{(t)}_{1,\mathbf{b}_1^{t-1}} \prod_{t<s\le p} \mathcal{L}^{(s)}_{1,\mathbf{b}_1^t} \cdot T^{\langle \mathbf{b}_1^{t-1}\rangle}_{b_t,\mathbf{b}^p_{t+1}} \dots T^{\langle \mathbf{b}_1^1\rangle}_{b_2,\mathbf{b}_3^p} T_{b_1,\mathbf{b}_2^p} \\
&= \mathcal{L}^{(t+2,p)}_{1,b_{t+1}} \cdot v^{(1,t)}_{b_{t+1},\mathbf{b}_1^t} \cdot \mathcal{L}^{(1,t-1)}_{1,b_t} T_{b_t,\mathbf{b}_1^{t-1}} \dots \mathcal{L}^{(1,1)}_{1,\mathbf{b}_2} T_{b_2,\mathbf{b}_1^1} \\
&\qquad \times \mathcal{L}^{(2)}_{1,b_1} \dots \mathcal{L}^{(t)}_{1,\mathbf{b}_1^{t-1}} \cdot T^{\langle \mathbf{b}_1^t\rangle}_{b_{t+1},\mathbf{b}^p_{t+2}} \cdot T^{\langle \mathbf{b}_1^{t-1}\rangle}_{b_t,\mathbf{b}^p_{t+1}} \dots T^{\langle \mathbf{b}_1^1\rangle}_{b_2,\mathbf{b}_3^p} T_{b_1,\mathbf{b}_2^p}
\end{aligned}$$

Therefore, by Lemma A1 we have

$$\begin{aligned}
Y_{t+1}\dots Y_1 &= v^{(1,t)}_{b_{t+1},\mathbf{b}_1^t} \cdot \mathcal{L}^{(t+2,p)}_{\mathbf{b}_1^t+1,\mathbf{b}_1^{t+1}} \cdot \mathcal{L}^{(1,t-1)}_{1,b_t} T_{b_t,\mathbf{b}_1^{t-1}} \dots \mathcal{L}^{(1,1)}_{1,\mathbf{b}_2} T_{b_2,\mathbf{b}_1^1} \\
&\qquad \times \mathcal{L}^{(2)}_{1,b_1} \dots \mathcal{L}^{(t)}_{1,\mathbf{b}_1^{t-1}} \cdot T^{\langle \mathbf{b}_1^t\rangle}_{b_{t+1},\mathbf{b}^p_{t+2}} \cdot T^{\langle \mathbf{b}_1^{t-1}\rangle}_{b_t,\mathbf{b}^p_{t+1}} \dots T^{\langle \mathbf{b}_1^1\rangle}_{b_2,\mathbf{b}_3^p} T_{b_1,\mathbf{b}_2^p} \\
&= \mathcal{L}^{(1,t)}_{1,b_{t+1}} T_{b_{t+1},\mathbf{b}_1^t} \dots \mathcal{L}^{(1,1)}_{1,\mathbf{b}_2} T_{b_2,\mathbf{b}_1^1} \mathcal{L}^{(2)}_{1,b_1} \dots \mathcal{L}^{(t+1)}_{1,\mathbf{b}_1^t} \prod_{t+1<s\le p} \mathcal{L}^{(s)}_{1,\mathbf{b}_1^{t+1},} \\
&\qquad \times T^{\langle \mathbf{b}_1^t\rangle}_{b_{t+1},\mathbf{b}^p_{t+2}} \dots T_{b_1,\mathbf{b}_2^p},
\end{aligned}$$

completing the proof of our claim. Taking $t = p$ in the claim completes the proof. $\square$

A3. **Proof of Lemma 2.34.** In this section we prove Lemma 2.34 and hence complete the proofs of all of our main results. Recall from section 2.6 that $\mathcal{H}^L_m$ is the R-submodule of $\mathscr{H}_{r,n}$ spanned by the elements

$$\{\, T_w L_1^{a_1} \dots L_{m-1}^{a_{m-1}} \mid 0 \le a_1, \dots, a_{m-1} < r \text{ and } w \in \mathfrak{S}_m \,\}.$$

To prove Lemma 2.34 we need the following result.

Lemma A5. *Suppose that a, b, k and l are positive integers such that $k \le l \le a$ and $1 \le s \le t \le p$. Then*

$$\mathcal{L}^{(s,t)}_{k,l} T_{a,b} = T_{a,b}\Big(\mathcal{L}^{(s,t)}_{b+k,b+l} + \sum_{m=b+k}^{b+l} \sum_{e=1}^{d(t-s+1)} h_{m,e} L_m^e\Big),$$

for some $h_{m,e} \in \mathcal{H}^L_m$.

Proof. For the duration of this proof let $L_{k,l}(Q) = \prod_{m=k}^{l}(L_m - Q)$, for $Q \in R$. Then $\mathcal{L}^{(s,t)}_{k,l} = \prod_{i=1}^d \prod_{u=s}^t L_{k,l}(\varepsilon^u Q_i)$. By the right handed version of [28, Lemma 5.6],

$$L_{k,l}(Q) T_{a,b} = T_{a,b}\Big(L_{b+k,b+l}(Q) + \sum_{m=b+k}^{b+l} h_m L_m\Big),$$

for some $h_m \in \mathcal{H}_m^L$. Therefore, there exist elements $h_{m,i,t} \in \mathcal{H}_m^L$ such that

$$\mathcal{L}_{k,l}^{(s,t)} T_{a,b} = T_{a,b} \prod_{i=1}^{d} \prod_{u=s}^{t} \Big(L_{b+k,b+l}(\varepsilon^u Q_i) + \sum_{m=b+k}^{b+l} h_{m,i,u} L_m \Big).$$

Collecting the terms in the product, we obtain $\mathcal{L}_{b+k,b+l}^{(s,t)}$, as the leading term, plus a linear combination of terms which are products of $d(t-s+1)$ elements, each of which is equal to either $L_{b+k,b+l}(\varepsilon^u Q_i)$ or $h_{m,i,u} L_m$, for some m, i, u as above. Expand the factors $L_{b+k,b+l}(\varepsilon^u Q_i)$ into a sum of monomials in $L_{b+k}, \dots, L_{b+l}$ and consider the resulting linear combination of products of these summands with the terms $h_{m,i,u} L_m$ above. Fix one of these products of $d(t-s+1)$ terms, say X, and let m be maximal such that L_m appears in X. By assumption the rightmost L_m which appears in X cannot have both T_m and T_{m-1} to its right, so using Lemma 2.8 we can rewrite X as a linear combination of terms of the form $h_{X,e} L_m^e$, where $1 \le e \le d(t-s+1)$ and $h_{X,e} \in \mathcal{H}_m^L$. Note that when we rewrite X in this form some of the $L_{m'}$, with $m' < m$, are changed into L_m when we move them to the right. However, T_m never appears to the right of these newly created L_m. The final exponent of L_m is at most $d(t-s+1)$ because no factor can increase the exponent of L_m by more than one. The result follows. □

Lemma A6. *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$. *Then*

$$v_{\mathbf{b}}^{+} = T_{\mathbf{b}'} \Big(\mathcal{L}^{(1)}_{\mathbf{b}_1^1+1,n} \mathcal{L}^{(2)}_{\mathbf{b}_1^2+1,n} \dots \mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n} + \sum_{l=1}^{p-1} \sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}} \sum_{e=1}^{dl} h_{l,m,e} L_m^e \Big)$$

for some $h_{l,m,e} \in \mathcal{H}_m^L$.

Proof. Recall that $v_{\mathbf{b}}^{+} = \mathcal{L}^{(1,p-1)}_{1,b_p} T_{b_p,\mathbf{b}_1^{p-1}} \mathcal{L}^{(1,p-2)}_{1,b_{p-1}} T_{b_{p-1},\mathbf{b}_1^{p-2}} \dots \mathcal{L}^{(1,1)}_{1,b_2} T_{b_2,\mathbf{b}_1^1}$. To prove the lemma let $v_{\mathbf{b},p}^{+} = 1$ and set $v_{\mathbf{b},k}^{+} = v_{\mathbf{b},k+1}^{+} \mathcal{L}^{(1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}_1^k}$, for $1 \le k < p$. We claim that if $1 \le k \le p$ then

$$v_{\mathbf{b},k}^{+} = T_{(b_p,\dots,b_{k+1},\mathbf{b}_1^k)} \Big(\mathcal{L}^{(1,p-1)}_{\mathbf{b}_1^{p-1}+1,\mathbf{b}_1^p} \dots \mathcal{L}^{(1,k)}_{\mathbf{b}_1^k+1,\mathbf{b}_1^{k+1}} + \sum_{l=k}^{p-1} \sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}} \sum_{e=1}^{dl} h'_{l,m,e} L_m^e \Big),$$

for some $h'_{l,m,e} \in \mathcal{H}_m^L$. When $k=p$ there is nothing to prove, so we may assume that $1 \le k < p$ and, by induction, that the claim is true for $v_{\mathbf{b},k+1}^{+}$. Therefore, by Lemma A5,

$$\begin{aligned} v_{\mathbf{b},k}^{+} = & T_{(b_p,\dots,b_{k+2},\mathbf{b}_1^{k+1})} \Big(\mathcal{L}^{(1,p-1)}_{\mathbf{b}_1^{p-1}+1,\mathbf{b}_1^p} \dots \mathcal{L}^{(1,k+1)}_{\mathbf{b}_1^{k+1}+1,\mathbf{b}_1^{k+2}} + \sum_{l=k+1}^{p-1} \sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}} \sum_{e=1}^{dl} h'_{l,m,e} L_m^e \Big) \\ & \times \quad T_{b_{k+1},\mathbf{b}_1^k} \Big(\mathcal{L}^{(1,k)}_{\mathbf{b}_1^k+1,\mathbf{b}_1^{k+1}} + \sum_{m=\mathbf{b}_1^k+1}^{\mathbf{b}_l^{k+1}} \sum_{e=1}^{dk} h''_{m,e} L_m^e \Big), \end{aligned}$$

for some $h'_{l,m,e}, h''_{m,e} \in \mathcal{H}_m^L$. Now, by Lemma 2.8, $T_{b_{k+1},\mathbf{b}_1^k}$ commutes with L_m whenever $m > \mathbf{b}_1^{k+1}$. Moreover, if $m > \mathbf{b}_1^{k+1}$ then

$$h'_{l,m,e} L_m^e T_{b_{k+1},\mathbf{b}_1^k} = h'_{l,m,e} T_{b_{k+1},\mathbf{b}_1^k} L_m^e = T_{b_{k+1},\mathbf{b}_1^k} h''_{l,m,e} L_m^e,$$

where $h''_{l,m,e} = T^{-1}_{b_{k+1},\mathbf{b}_1^k} h'_{l,m,e} T_{b_{k+1},\mathbf{b}_1^k}$. It is easy to check that $h''_{l,m,e} \in \mathcal{H}^L_m$. Next note that $T_{(b_p,\dots,b_{k+2},\mathbf{b}_1^{k+1})} T_{b_{k+1},\mathbf{b}_1^k} = T_{(b_p,\dots,b_{k+1},\mathbf{b}_1^k)}$. Therefore, $v^+_{\mathbf{b},k}$ is equal to

$$v^+_{\mathbf{b},k} = T_{(b_p,\dots,b_{k+1},\mathbf{b}_1^k)} \Big(\mathcal{L}^{(1,p-1)}_{\mathbf{b}_1^{p-1}+1,\mathbf{b}_1^p} \dots \mathcal{L}^{(1,k+1)}_{\mathbf{b}_1^{k+1}+1,\mathbf{b}_1^{k+2}} + \sum_{l=k+1}^{p-1} \sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}} \sum_{e=1}^{dl} h''_{l,m,e} L_m^e \Big) \times \Big(\mathcal{L}^{(1,k)}_{\mathbf{b}_1^k+1,\mathbf{b}_1^{k+1}} + \sum_{m=\mathbf{b}_1^k+1}^{\mathbf{b}_1^{k+1}} \sum_{e=1}^{dk} h''_{m,e} L_m^e \Big).$$

To complete the proof of the claim observe that

$$\mathcal{L}^{(1,p-1)}_{\mathbf{b}_1^{p-1}+1,\mathbf{b}_1^p} \dots \mathcal{L}^{(1,k+1)}_{\mathbf{b}_1^{k+1}+1,\mathbf{b}_1^{k+2}} = \mathcal{L}^{(1)}_{\mathbf{b}_1^{k+1}+1,n} \dots \mathcal{L}^{(k+1)}_{\mathbf{b}_1^{k+1}+1,n} \mathcal{L}^{(k+2)}_{\mathbf{b}_1^{k+2}+1,n} \dots \mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n}.$$

Therefore, when we write this element as a polynomial in $L_{\mathbf{b}_1^{k+1}+1},\dots,L_n$, the exponent of L_m is at most dl if $\mathbf{b}_1^l < m \le \mathbf{b}_1^{l+1}$ for some $k+1 \le l \le p-1$. Using this observation it is now a straightforward exercise to expand the formula for $v^+_{\mathbf{b},k}$ above and show that $v^+_{\mathbf{b},k}$ can be written in the required form, thus completing the proof of the claim.

Returning to the proof of the lemma, observe that $v^+_{\mathbf{b}} = v^+_{\mathbf{b},1}$ and that the statement of the lemma is the special case of the claim above when $k = 1$ (and setting $k = 0$ in the last displayed equation). $\square$

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